MATH 23B PROBLEM SET 3

due February 25th

Let V be a vector space,$\mathcal{B}^r(V)$ the space of r-linear forms on V and $\Omega^r(V) \subset \mathcal{B}^r(V)$ the subspace of antisymmetric forms. For any $\sigma \in S_r$ we denote by $\hat{\sigma}$ the linear map from $\hat{\sigma} : \mathcal{B}^r(V) \to \mathcal{B}^r(V)$ given by

$$\hat{\sigma}(\omega)(v_1, ..., v_r) := \omega(v_{\sigma^{-1}(1)}, ..., v_{\sigma^{-1}(r)})$$

.

1.a) Show that $\Omega^r(V) = \{\omega \in \mathcal{B}^r(V) | \hat{\sigma}(\omega) = sign(\sigma)\omega \,\forall \sigma \in S_r$.

b) Show that for any $\sigma', \sigma'' \in S_r$ we have $\hat{\sigma} = \hat{\sigma}' \circ \hat{\sigma}''$ where $\sigma := \sigma' \circ \sigma''$.

Let $Alt : \mathcal{B}^r(V) \to \mathcal{B}^r(V)$ be a linear operator $Alt := 1/r! \sum_{\sigma \in S_r} sign(\sigma)\hat{\sigma}$.

c) Show that $Alt^2 = Alt$

d) Show that $Im(Alt) = \Omega^r(V)$ and that $Alt(\omega) = \omega \,\forall \omega \in \Omega^r(V)$.

Let $T : V \to V$ be a linear operator, $B' = (v'_1, ..., v'_d)$ and $B'' = (v''_1, ..., v''_d)$ two bases of V. Let $A' = (a'_{i,j})$ and $A'' = (a''_{i,j})$ be the matricies of A in respect bases B' and B''. Let $C = (c_{i,j})$ be the matrix such that $v''_j = \sum_i c_{i,j} v'_i$

2. a) How to find the matrix A'' if you know matricies A' and C?

b)Show that $Det(A') = Det(A'')$ using the expression for A'' from 2 a).

3. Let A be an antisymmetric $n \times n$ matrix [that is $a_{ij} = -a_{ji}$]. Show that $Det(A) = 0$ if n is odd.

Hint. Start with the case $n = 3$

4. Let $A = \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ and $B = \begin{bmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{bmatrix}$

Show that $Det(B) = 2Det(A)$.

Let $A = \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & 1-x & 1 & ... & 1 \\ 1 & 1 & 2-x & ... & 1 \\ . & . & . & ... & (n-1)-x \end{bmatrix}$, $F(x) := Det(A)$.

5. a)Find zeros of the function $F(x)$.

b)Compute $F(x)$.

Let $A^{\{k\}}, 1 \le k \le n$ be the $n \times n$ matrix $A^{\{k\}} = (a_{ij}^{\{k\}})$ where $a_{ij}^{\{k\}} = x_j^{i-1}$ if $i < k$ and $a_{ij}^{\{k\}} = x_j^i$ if $i \ge k$.

6. a) Show that $Det(A^{\{k\}})$ is a polinomial $F^{\{k\}}$ in $x_i, 1 \le i \le n$ which can be written in the form $F^{\{k\}} = Q^{\{k\}} R$ where $R(x_1, ..., x_n) :=$

$\prod_{1\le i<j\le n}(x_j - x_i)$ and $Q^{\{k\}} = Q^{\{k\}}((x_1, ..., x_n)$ is a symmetric polynomial.

Find the polynomila $Q^{\{k\}}((x_1, ..., x_n)$.

A hint. Consider the case $n = 3$ in details.