MATH 23B SOLUTION SET 1

1.a,b) You all seemed to do fine on these.

1.c) This follows from 1.d) because $Alt(\omega)$ is anti-symmetric and is therefore fixed by Alt.

1.d) We need to prove two statements, firstly that $Alt(\omega)$ is anti-symmetric for any polylinear form , and secondly that if ω is anti-symmetric then Alt fixes ω. The first fact proves that $Im(Alt) \subseteq \Omega^r(V)$ and the second that $Im(Alt) \supseteq \Omega^r(V)$.

$$\hat{\sigma'} Alt(\omega) = 1/r! \sum_{\sigma \in S_r} sign(\sigma)\hat{\sigma'}(\sigma)\hat{\omega}$$

. Let $\sigma'' = \sigma'\sigma$. Since permutations are invertable we can sum over $sigma''$. That is, by 1.b)

$$\hat{\sigma'} Alt(\omega) = 1/r! \sum_{\sigma'' \in S_r} sign(\sigma')^{-1} sign(\sigma'')\hat{\sigma''}\omega = sign(\sigma') Alt(\omega)$$

. (Where we have used the fact that $sign(\sigma')^{-1} = sign(\sigma')$.) Therefore, by 1.a) $Alt(\omega)$ is anti-symmetric. If ω is anti-symmetric, then

$$Alt(\omega) = 1/r! \sum_{\sigma \in S_r} sign(\sigma)(\sigma)\hat{\omega}$$

. By 1.a) again, we see that

$$Alt(\omega) = 1/r! \sum_{\sigma \in S_r} sign(\sigma) sign(\sigma)\omega$$

. But since $(\pm 1)^2 = 1$ and S_n has $n!$ elements, $Alt(\omega) = r!/r!\omega = \omega$. So we have proved what we wanted.

2.a) Applying A'' is the same as applying C^{-1} (to get you a vector in the basis B'), then applying A' and then applying C (to put it back in the basis B''). That is to say, $A'' = CA'C^{-1}$. (Not the other way around as a lot of you seem to have done.)

2.b)

$$|A''| = |CA'C^{-1}| = |C||A'||C^{-1}| = |C||C^{-1}||A'| = |CC^{-1}A'| = |A'|$$

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3. Since A is anti-symmetric, $A^T = -A$. It is easy to see that if n is odd then $|-A| = -|A|$ (for example we are multiplying each row by -1 and there are n rows). But we know from class $|A^T| = |A|$. Hence, $|A| = |A^T| = |-A| = -|A|$. Therefore $|A| = 0$.

4. Let C be the 3x3 matrix with zeros on the diagonal and ones elsewhere. Then we notice that $B = AC$. So, $|B| = |AC| = |A||C| = 2|A|$. (Of course there are many other solutions that work but this is definitely the fastest.)

(Notice that both determinants are anti-symmetric tri-linear forms over a three-dimensional vector space. So they must be some multiple of each other.)

5. Subtract the first row from all the other rows. As discussed at some point this does not change the determinant. The new matrix is upper diagonal and so

the determinant is the product of the elements on the diagonal. Hence, $F(x) = \prod_{i=0}^{n-2}(i-x)$. Therefore the zeroes of F are at $0, 1, ..., n-2$.

6.a) The fact that R divides $F^{\{k\}}$ is proved exactly the same way as the other question we did in class: when $x_i = x_j$ F is zero, so $(x_i - x_j)|F^{\{k\}}$. Just as in class this implies that $R|F^k$. Since R and $F^{\{k\}}$ are anti-symmetric, $Q^{\{k\}}$ must be symmetric.

6.b) The degree of $Q^{\{k\}}$ is $n+k-1$ (subtract the degrees of $F^{\{k\}}$ and R). But this is not enough to find $Q^{\{k\}}$. Furthermore, $Q^{\{k\}}$ is linear in each individual variable. We also notice that every term in $F^{\{k\}}$ and R has the same degree. It is easy to show that this implies that every term in $Q^{\{k\}}$ must have the same degree (any term with a smaller or larger degree would contribute a term with a different degree to $F^{\{k\}}$). Hence for some constants $c_{(i_1,\ldots i_{n-k+1})}$,

$$Q^{\{k\}} = \sum_{1\le i_1<i_2<\ldots<i_{n-k+1}\le n} c_{(i_1,\ldots i_{n-k+1})} \prod_{\ell=1}^{n-k+1} x_{i_\ell}$$

. Now we need only find the constants. But if you multiply this expression for $Q^{\{k\}}$ by R (I'm not going to write it out here because it'd take a while to type out) you see that the terms in $F^{\{k\}}$ appear only once (while other terms may appear several times). Furthermore if you look at these terms you will see that all the c's must in fact be 1 (the other terms are more complicated, but we don't have to show that they actually cancel out because we've already shown that some $Q^{\{k\}}$ exists and now we've shown that it must have a certain form, so we don't need to show that this expression actually works). Therefore,

$$Q^{\{k\}} = \sum_{1\le i_1<i_2<\ldots<i_{n-k+1}\le n} \prod_{\ell=1}^{n-k+1} x_{i_\ell}$$

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