Let $f : [0,1] \to \mathbb{R}$ be the function such that $f(x) = 1/q$ if $x = p/q$ where p, q are relatively prime and $f(x) = 0$ if x is irrational.

1. a)For which $x \in [0,1]$ the function f is continuous at x?

b)Is f integrable?

2. Let f, g be bounded functions on the square $S = [0,1] \times [0,1]$.

a)Show that for any partition P of S we have $U(P, f+g) \leq U(P, f) + U(P, g)$ and $L(P, f+g) \geq L(P, f) + L(P, g)$

b)Show that the function $f + g$ on S is integrable if f and g are integrable.

3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $0 \leq f(x) \leq 1, x \in [0,1]$. Define a function F on $S = [0,1] \times [0,1]$ by $F(x,y) = 1$ for $y \leq f(x)$ and $F(x,y) = 0$ for $y > f(x)$.

a)Show that F is integrable.

b)Prove that $\int_S F(x,y)dxdy = \int_0^1 f(x)dx$

Let X be a bounded subset of $\mathbb{R}^n$. Choose a rectangle $S \in \mathbb{R}^n$ which contains X and denote by $\chi_X^S : S \to \mathbb{R}$ the functions such that $\chi_X(s) = 1$ if $s \in X$ and $\chi_X^S(s) = 0$ if $s \in S - X$. We say that the set X is *measurable* if the function χ_X^S is integrable. In this case we define $vol(X)^S := \int_S \chi_X^S$.

4.a) Show that if for some rectangle $S \supset X$ the function χ_X^S is integrable then for any other rectangle $S' \supset X$ the function $\chi_X^{S'}$ is also integrable and $vol(X)^S = vol(X)^{S'}$.

Remark. This result shows that we can define the volume $vol(X)$ for any measurable set X.

b) Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator,$U = [0,1] \times [0,1]$ and $X = A(U)$. Show that $vol(X) = |Det(A)|$.

c)Formulate and prove the analogous result for a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$.

A hint. First analize the case when A is a linear operator of one of the following 3 types.

1)$A(e_i) = e_i, i \neq j, A(e_j) = ae_j$ for some $a \in \mathbb{R} - 0, 1 \leq j \leq n$

2)$A(e_i) = e_i, i \neq j, A(e_j) = e_j + ce_k$ for some $1 \leq j,k \leq n, c \in \mathbb{R}$

3)$A(e_i) = e_i, i \neq j,k, A(e_j) = e_k.A(e_k) = e_j$ for some $1 \leq j,k \leq n$