1. a) f is continuous at all irrational points. To see this, choose x_0 an irrational and $\epsilon > 0$. Then we can choose an integer $n > 1/\epsilon$, so that $1/n < \epsilon$. The set of elements $X = \{x \in [0,1] | f(x) \geq 1/n\}$ is just the set of fractions between 0 and 1 with denominators less than n, hence it is finite. So we can choose a $\delta > 0$ s.t. no elements of X are within δ of x_0, namely $\delta < min(x_0 - X)$. Thus we have a δ s.t. $\forall x$ s.t. $|x_0 - x| < \delta$, $|f(x_0) - f(x)| = f(x) < 1/n < \epsilon$.

f is discontinuous at all rational points. For a proof, choose $x_0 = p/q$ (in lowest terms) rational and $\epsilon = \frac{1}{2q}$. For all $\delta > 0$, there is an irrational x within δ of x_0. For this x, $|f(x_0) - f(x)| = f(x_0) = \frac{1}{q} > \frac{1}{2q}$, hence f is discontinuous at x_0.

b) f is integrable. To show this, we need to prove that $\forall\ \epsilon > 0$, $\exists$ a partition P s.t. $U(P,f) - L(P,f) < \epsilon$. Since any segment $[x_{i-1}, x_i]$ of the real line contains an irrational, $inf(f)$ is 0 on every segment. Hence $L(P,f) = \sum_{i=1}^{n} (x_i - x_{i-1}) inf_{[x_{i-1},x_i]}(f) = 0\ \forall$ P. So if the integral exists, its value is 0. To show that $U(P,f)$ can be small, choose some integer n and some $\delta > 0$ Then let the partition $P_n(\delta)$ have segments of length δ surrounding all k_n rational numbers p/q with $q \leq n$. Then, for this partition, we can bound the elements of the sum that involve the δ-segments by $1(x_i - x_{i-1}) = \delta$ and the other elements by $\frac{1}{n}(x_i - x_{i-1})$:

$$\begin{aligned} U(P_n(\delta), f) &= \sum_i (x_i - x_{i-1}) sup_{[x_{i-1},x_i]}(f) \\ &< k_n \delta + \frac{1}{n}. \end{aligned}$$

We can make this last expression less than any given epsilon by choosing n s.t. $1/n < \epsilon$ and then choosing δ sufficiently small. Hence f is integrable and $\int_0^1 f(x)dx = 0$.

Many people noted that f is discontinuous on a set of measure 0 and therefore is integrable. Unfortunately, we hadn't covered this material at the time of this problem set (some related material is on problem set 5), so only partial credit was given for that solution. In fact, the method used to prove this problem is very similar to a proof that establishes this theorem.

2. a) We need the following lemma.

Lemma: For any two sets A, B of real numbers, define A+B as $\{a + b | a \in A, b \in B\}$. Then $sup(A) + sup(B) \geq sup(A + B)$ and $inf(A) + inf(B) \leq inf(A + B)$.

Note: Why is this hard, you ask? Well, the problem is that $sup(A)$ is not necessarily achieved on A, for example if $A = (0,1)$, then $sup(A) = 1$, but 1 is not in A. This counterexample is not possible if A is compact and $sup(A)$ is achieved. Why? We will prove the general case, because it's not hard and informative. As it turns out, we can also strengthen this to say that $sup(A)+sup(B) = sup(A+B)$, but we don't need this, and the method is no different from the following proof of the Lemma.

Proof: We prove only the first statement, the second is in all ways similar. Assume $\exists$ A,B s.t. $sup(A) + sup(B) < sup(A + B)$. So $\exists a + b \in A + B$ s.t. $a + b > sup(A) + sup(B)$ (otherwise sup(A+B) would be smaller). But since necessarily $a \leq sup(A)$ and $b \leq sup(B)$, this is a contradiction.

Now, for any set $A \subset [0,1]\times[0,1]$, $f(A)$ and $g(A)$ are sets of real numbers and $(f+g)(A) \subset f(A)+f(B)$ so $sup_A(f+g) = sup((f+g)(A)) \leq sup(f(A)+f(B))$. Hence, by the Lemma, $sup_A(f)+sup_A(g) \geq sup_A(f+g)$

$$\begin{aligned} U(P, f+g) &= \sum_{i,j}(x_i - x_{i-1})(y_j - y_{j-1})sup_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(f+g) \\ &\leq \sum_{i,j}(x_i - x_{i-1})(y_j - y_{j-1})sup_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(f) \\ &+ sup_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(g) \\ &= U(P,f) + U(P,g). \end{aligned}$$

The inequality with $L(P, f+g)$ is completely analogous.

b) By part a, $\forall$ f, g: $[0,1]\times[0,1] \to \mathbb{R}$ integrable and P a partition of $[0,1]\times[0,1]$,

$$U(P, f+g) - L(P, f+g) \leq U(P,f) - L(P,f) + U(P,g) - L(P,f).$$

Thus for any $\epsilon > 0$, we simply choose P_1 s.t. $U(P_1, f) - L(P_1, f) < \epsilon/2$ and P_2 s.t. $U(P_2, g) - L(P_2, g) < \epsilon/2$. Then we can take P a partition that is simultaneously a refinement of P_1 and P_2 and get $U(P,f) \leq U(P_1,f)$, $U(P,g) \leq U(P_2,g)$, $L(P,f) \geq L(P_1,f)$, and $L(P,g) \geq L(P_2,g)$. So

$$U(P,f) - L(P,f) \leq U(P_1,f) - L(P_1,f) < \epsilon/2$$

$$U(P,g) - L(P,g) \leq U(P_2,g) - L(P_2,g) < \epsilon/2$$

and

$$U(P, f+g) - L(P, f+g) \leq U(P,f) - L(P,f) + U(P,g) - L(P,f) < \epsilon$$

for any $\epsilon > 0$. Hence $f+g$ is integrable.

3. a) This can be seen as the analogue of a one-dimensional piecewise continuous function, but as in the one-dimensional case where we

allowed a finite (or at least countably infinite) number of discontinuities, there are constraints on what the set of discontinuities can look like. In this case, making the function $f : [0,1] \to [0,1]$ continuous is enough.

So to prove this, we need to take a partition of $[0,1]$ and turn it into one of $[0,1] \times [0,1]$. To this end, let P' be the partition $0 = x_0 < x_1 < \ldots < x_n = 1$. Then consider a partition Q of $[0,1]$ containing the points $sup[x_{i-1}, x_i](f), inf[x_{i-1}, x_i](f)$ $\forall i$. We can use these two, P' and Q, to partition $[0,1] \times [0,1]$.

For any partition, it is clear that on any subrectangle that does not contain an element of the form $(x, f(x))$, the value of F is constant. (This can be made rigorous by using convexity to look at a line joining points on which F is 0 and 1 and then applying the mean value theorem to an expression of the form $y - f(x)$.) Hence on these subrectangles, sup(F) = inf(F), so when we look at the difference $U(P,F) - L(P,F)$, the only terms in the sums which do not cancel are those which involve subrectangles containing some point of the form $(x, f(x))$, and on these $sup(F) = 1$ and $inf(F) = 0$:

$$\begin{aligned} U(P,F) - L(P,F) &= \sum_{\text{some}||i,j} (x_i - x_{i-1})(y_j - y_{j-1})(sup_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(F) \\ &- inf_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(F)) \\ &= \sum_i (x_i - x_{i-1})(y_{k^+(i)} - y_{k^-(i)})(sup_{[x_{i-1},x_i]\times[y_{k^-(i)},y_{k^+(i)}]}(F) \\ &- inf_{[x_{i-1},x_i]\times[y_{k^-(i)},y_{k^+(i)}]}(F)) \\ &= \sum_i (x_i - x_{i-1})(y_{k^+(i)} - y_{k^-(i)}). \end{aligned}$$

where $y_{k^-(i)}$ and $y_{k^+(i)}$ are the bottom and top limits of the subrectangles which contain elements of the form $(x, f(x))$ for a given segment $[x_{i-1}, x_i]$. Notice that all of these subrectangles are contiguous in the y coordinate since f is a single valued function.

Now if we use $P = P' \times Q$ to partition $[0,1] \times [0,1]$, we notice that the values $y_{k^-(i)} = inf_{[x_{i-1},x_i]}(f)$ and $y_{k^+(i)} = sup_{[x_{i-1},x_i]}(f)$. (This is not entirely true in our current formulation because of nasty boundary cases. If a point of the form $(x, f(x))$ occurs only on the boundary of some subrectangle, nasty things can occur. To deal with this, we should actually define Q so that it contains the points $sup_{[x_{i-1},x_i]}(f) + \eta, inf_{[x_{i-1},x_i]}(f) + \eta$ $\forall i$ and some $\eta > 0$ which we would then make small. However, the current argument makes the idea more clear without mucking around in *too* many details.) So we are left with

$$U((P',Q),F)-L((P',Q),F) < \sum_i (x_i-x_{i-1})(sup_{[x_{i-1},x_i]}(f)-inf_{[x_{i-1},x_i]}(f)).$$

which is just $U(P',f)-L(P',f)$. Since f is integrable, this can be made less than any ϵ and hence so can $U(P,F)-L(P,F)$. So F is integrable.

b) Using the machinery of the previous section, we see that using the partition $P = P' \times Q$ gives us

$$\begin{aligned} U(P,F) &= \sum_i (x_i - x_{i-1}) inf_{[x_{i-1},x_i]}(f) \\ &+ \sum_i (x_i - x_{i-1})(sup_{[x_{i-1},x_i]}(f) - inf_{[x_{i-1},x_i]}(f)) \\ &= \sum_i (x_i - x_{i-1})(sup_{[x_{i-1},x_i]}(f) \\ &= U(P',f) \end{aligned}$$

and similarly for $L(P,F)$. Hence $\int_{[0,1]\times[0,1]} F = \int_{[0,1]} f$.

4. a) At first glance, this seems obvious; the difficulty arises when we realize that we can only integrate on rectangles. This problem comes very close to proving that if an integral on a bounded set exists, it is well-defined. It does the case of $f(x) = 1$, which is only marginally easier than the general case.

First note that for rectangles S and S', $S \cap S'$ is a rectangle and that $\chi_X^S = \chi_X^{S'}$ on $S \cap S'$ (they *restrict* to the same function on $S \cap S'$). For any given partition P of $S \cap S'$, we can extend it to a partition P^S of S and a partition $P^{S'}$ of S'.