Let $X \subset \mathbb{R}^N$ be a set and $f_n : X \to \mathbb{R}$ be a sequence of functions. We say that the sequence f_m *converges uniformally* to a function f on X if for any $\epsilon > 0$ there exists a number n_0 such that for any $n > n_0$ and any $x \in X$ we have $|f(x) - f_n(x)| < \epsilon$.

1. Let a $f_n : X \to \mathbb{R}$ be a sequence of continuous functions which converges uniformally to a function f.

a) Show that the function f is continuous.

b)Let X be a rectangle. Show that the sequence of numbers $I_n := \int_X f_n(x)dx_1...dx_N$ is convergent to $\int_X f(x)dx_1...dx_N$.

c)Construct a sequence of continuous functions f_n on $[0,1]$ such that $\int_0^1 f_n(x)dx = 1$ but for any $x \in [0,1]$ we have $\lim_{n\to\infty} f_n(x) = 0$.

Let $X \subset \mathbb{R}^N$ be a compact subset and $U_a \subset \mathbb{R}^N, a \in A$ be a open sets such that $X \subset \bigcup U_a$.

Definition. For and $x \in X$ we denote by $M_x \subset \mathbb{R}$ the set of real numbers $r > 0$ such $B_r(x) \subset U_a$ for some $a \in A$ where $B_r(x)$ is a ball of radius r with the center at x. Let $r(x) := \sup M_x$.

2. a) Prove that the function $x \mapsto r(x)$ on X is continuous.

b)Show that there exist a *finite* subset $a_1, ..., a_n \subset A$ such that $X \subset \bigcup U_{a_i}, 1 \leq i \leq n$.

Let X be as before and f be a bounded real-valued function on X.

3. Show that f is integrable if and only for any $\epsilon > 0$ there exists a partition P of X such that $U(f,P) - L(f,P) < \epsilon$.

Let X and f be as before. For any point $x \in X$ and $r > 0$ we define $o(f,x,r) = \sup |f(y) - f(x)|$ for $y \in X, |y - x| < r$ and $o(f,x) = \inf o(f,x,r)$ for $r > 0$. We call the number $o(f,x,r)$ *the oscillation* of f at x. For any $a \in \mathbb{R}_>$ we define $X_a = \{x \in X | o(f,x) \geq a\}$.

4. a) Show that f is continuous at a point $x \in X$ if and only if $o(f,x) = 0$.

b) Prove that for any $a \in \mathbb{R}_>$ the subset $X_a \subset \mathbb{R}^N$ is closed.

c) Show that if f is integrable then for any $a \in \mathbb{R}_>$ and any $\epsilon > 0$ we can cover the set X_a by rectangles with the total volume less then ϵ.

d)Formulate and prove the converse theorem.

e) Show that for an bounded integrable function f on $[0,1]$ and any $\epsilon > 0$ there exists $\delta > 0$ such that for any partition $P = \{0 = x_0 < ....x_n = 1\}$ such that $x_i - x_{i-1} < \delta \forall i, 1 \leq i \leq n$ we have $U(f,P) - L(f,P) < \epsilon$.