MATH 23B SOLUTION SET 5

1.a,b Everyone got these question pretty much so I'm not going to waste my time on it.

1.c A few examples are:

$$f_n(x) = \begin{cases} 4n^2x & \text{if } 0 \le x \le 1/2n, \\ -4n^2x + 4n & \text{if } 1/2n < x \le 1/n, \\ 0 & \text{if } 1/n < x \le 1. \end{cases}$$

$$f_n(x) = (n+1)(n+2)(1-x)x^n$$

$$f_n(x) = (1/n)x^{\frac{x}{n}}(\log x + 1)$$

or if you're completely insane:

$$f_n(x) = \frac{2(k+1)}{\pi}\cos^k(\frac{\pi x}{2})\sin(\frac{\pi x}{2})$$

2.a) Suppose that r is discontinuous at $x_0 \in X$ then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x_\delta$ s.t. $|x_\delta - x_0| < \delta$ and $|r(x_\delta) - r(x_0)| > \varepsilon$. Now we have two cases, either $r(x_\delta) < r(x_0)$ or vice-versa. Both cases are exactly the same so i'll only do the first case. Since $|r(x_\delta) - r(x_0)| > \varepsilon$ we have that $r(x_\varepsilon) + \varepsilon < r(x_0)$. But then, $B_{r(x_0)}(x_0) \subsetneqq B_{r(x_\varepsilon)}(\varepsilon) \subset U_{a_i}$. This contradicts the definition of $r(x_0)$. Therefore, r is continuous.

b) I'll do two proofs. Firstly a correct version of the proof that most of you attempted and secondly the proof that uses 2.a).

Proof 1: Assume that there does not exist a finite subcover. Take some hypercube C_0 with sides of length ℓ containing X. Cut this cube into 2^n smaller cubes. Take the intersection of each of these cubes with X. At least one of these intersections does not have a finite subcover. Call this smaller cube C_1. Continue choosing cubes C_i such that $C_i \cap X = X_i$ has no finite subcover. Now the X_i are nested compact sets, so by some theorem we've proved in class (methinks it's called the nested balls theorem) there is some point p contained in all of them. This point p is contained in some open set U_a. This open set contains an open ball of some radius r around p. Since all the X_i's are boxes containing p and have length $\frac{\ell}{2^i}$ for $i > \log_2(\ell/r)$ we have that $X_i \subset U_a$ contradicting the assumption that X_i had no finite subcover.

Proof 2: By the convergent subsequence definition the image of a compact set under a continuous map is still compact. Therefore the image of X under r is compact. In particular it contains its inf. Since $r(x) \neq 0$ the inf is non-zero, therefore for some $\varepsilon > 0$ $r(x) \ge \varepsilon$. So, every single point is contained in a ball of radius ε. Now we need only show that any bounded set in $\mathbb{R}^n$ can be covered by finitely many balls with centers in the set of radius ε for any ε. Since the ball of radius ε about any point contains any ball of radius $\varepsilon/4$ containing that point we need only show that any bounded set can be covered by finitely many balls with centers inside or outside the set with radius ε for any ε. (A set with this property is called totally bounded.) It is blatantly obvious that every bounded set in $\mathbb{R}^n$ is totally bounded (just take a cube containing the set and cut it up into cubes inscribed in the ball of radius ε). Therefore we can choose finitely many balls of radius ε about points in X which cover X and since each of these balls in contained in one of our open sets we can find an open subcover.

3. First we show that if f is integrable then there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$ for any ε. We know that since f is integrable $\inf(U(f,P)) = \sup(L(f,P))$. Therefore, there exist partitions P_1 and P_2 such that $U(f,P_1) - L(f,P_2) < \varepsilon$ for any ε. Now take $P_3 = P_1 \cup P_2$ a finer partition than both of the others. Now, $U(f,P_3) - L(f,P_3) < U(f,P_1) - L(f,P_2) < \varepsilon$ as we had hoped to show. Secondly we show the opposite direction. This is much easier (its important to notice immediately which direction of a question is difficult because usually one of them is trivial). $\inf(U(f,P)) - \sup(L(f,P)) < U(f,P)) - L(f,P) < \varepsilon$ for any ε. Therefore, $\inf(U(f,P)) - \sup(L(f,P)) = 0$ and f is integrable.

4.a) Suppose that f is continuous at a. Then for every number $\varepsilon > 0$ there exists some δ such that for $x \in B_\delta(a)$ we have that $|f(x) - f(a)| < \varepsilon$. Therefore, $o(f,a,r) < 2\varepsilon$ for any ε. Therefore, $o(f,a) = 0$. Now we prove the other direction. If $o(f,x) = 0$ then for any ε, there exists δ such $o(f,a,\delta) < \varepsilon$. So for any x such that $|a - x| < \delta$ $|f(a) - f(x) < \varepsilon$ which is exactly the definition of continuity.

b) X_a is closed iff $\mathbb{R}^n - X_a$ is open. But, $\mathbb{R}^n - X_a = (\mathbb{R}^n - X) \cup (X - X_a)$. Since X is closed we need only show that $(X - X_a)$ is open. So, take $x \in (X - X_a)$. Then by definition, $o(f,x) < a$. Therefore, there exists $r > 0$ such that $o(f,x,r) < a$. So consider $B_r(x)$. For any $y \in B_r(x)$ we have that $B_{(r-|x-y|)}(y) \in B_r(x)$. Therefore, $o(f,y) < a$. Hence $B_r(x) \in (X - X_a)$. Therefore, $X - X_a$ is open.

c,d,e) these are Spivak Theorems 3.8. and 3.7, so I will not repeat them here since you can look that up in someone's copy of the book or in the library.