

1. Let f be a function on \mathbb{R} with compact support and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ a continuously differentiable function such that $\phi(x) \rightarrow \pm\infty$ for $x \rightarrow \pm\infty$.

a) Prove that the function $f \circ \phi : \mathbb{R} \rightarrow \mathbb{R}$ is also a function with compact support.

b) Show that in the case when f is continuous we have

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f \circ \phi(y)\phi'(y)dy$$

c) Show that for any integrable function f the function $f \circ \phi$ is also integrable and

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f \circ \phi(y)\phi'(y)dy$$

2. Let $f(x)$ be a continuously differentiable function on $[-1, 1]$. For any $\epsilon > 0$ we define

$$I_{\epsilon}(f) := \int_{-1}^1 \frac{f(x)}{x + i\epsilon} dx, i := \sqrt{-1}$$

In other words

$$I_{\epsilon}(f) := \int_{-1}^1 \frac{f(x)(x - i\epsilon)}{x^2 + \epsilon^2} dx = \int_{-1}^1 \frac{f(x)x}{x^2 + \epsilon^2} dx - i\epsilon \int_{-1}^1 \frac{f(x)}{x^2 + \epsilon^2} dx$$

We also define

$$I_{\epsilon}^{-}(f) := \int_{-1}^1 \frac{f(x)}{x - i\epsilon} dx$$

a) Show that in the case when $f(0) = 0$ the integral $\int_{-1}^1 f(x)/x$ exists and is equal to $\lim_{\epsilon \rightarrow 0} I_{\epsilon}(f)$ and to $\lim_{\epsilon \rightarrow 0} I_{\epsilon}^{-}(f)$.

b) Show that the existence of limits $I_{+}(f) := \lim_{\epsilon \rightarrow 0} I_{\epsilon}(f)$ and $I_{-}(f) := \lim_{\epsilon \rightarrow 0} I_{\epsilon}^{-}(f)$, $\epsilon > 0$ for any continuously differentiable function f on $[-1, 1]$.

c) Show that $I_{+}(f) - I_{-}(f) = cf(0)$, $c \in \mathbb{C}$ and find c .

Let $I = [a_1, b_1] \times [a_2, b_2] \dots \times [a_n, b_n] \subset \mathbb{R}^n$ be a rectangle. For any i , $1 \leq i \leq n$ we denote by I_i the rectangle

$$I_i = [a_1, b_1] \times \dots \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times [a_n, b_n] \subset \mathbb{R}^{n-1}$$

For any continuous function f on I and any i , $1 \leq i \leq n$ we define functions f_i^{\pm} on the rectangle I_i by the rule

$$f_i^{+}(x_1, \dots, x_{n-1}) := f(x_1, \dots, x_{i-1}, b_i, x_i, \dots, x_{n-1}) \text{ and}$$

$$f_i^{-}(x_1, \dots, x_{n-1}) := f(x_1, \dots, x_{i-1}, a_i, x_i, \dots, x_{n-1}).$$

3 a) [Green's Theorem]

Let P_i , $1 \leq i \leq n$ be continuously differentiable functions on I .

Show that

$$\int_I \sum_{i=1}^n (-1)^{i-1} \partial P_i / \partial x_i (x_1, \dots, x_n) dx_1 \dots dx_n = \sum_{i=1}^n (-1)^{i-1} \int_{I_i} P_i^+(x_1, \dots, x_{n-1}) - P_i^-(x_1, \dots, x_{n-1}) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n$$

b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function such that $\text{supp}(f) \subset I$ and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuously differentiable function such that $\phi(x) = x$ for $x \in \mathbb{R}^n - I$.

Show that $\int_I f(x) dx_1 \dots dx_n = \int_I \tilde{f}(y) dy_1 \dots dy_n$ where $\tilde{f}(y) := f(\phi(y)) J(\phi)(y)$, $J(\phi)(y) := \text{Det}(D(\phi)(y))$