MATH 23B: SOLUTIONS TO PROBLEM SET 6

1) a. Since f has compact support, $\exists a, b \in \mathbb{R}$ s.t. f is zero outside of $[a, b]$. Now $\phi(x) \to \pm\infty$ for $x \to \infty$ means that $\exists x_b \in \mathbb{R}$ s.t. $\forall x > x_b$, $\phi(x) > b$ and similarly, $\exists x_a \in \mathbb{R}$ s.t. $\forall x > x_a$, $\phi(x) < a$. Thus $[x_a, x_b]$ is a compact support for $f \circ \phi$.

b. Since f, ϕ, and ϕ' are all continuous, $f \circ \phi(x)\phi'(x)$ is continuous. Since $f \circ \phi(x)$ has compact support, $f \circ \phi(x)\phi'(x)$ has compact support. Hence $f \circ \phi(x)\phi'(x)$ is integrable from $-\infty to \infty$.

Now we define $F(x) = \int_a^x f(x)dx$, so that $F'(x) = f(x)$ by the Fundamental Theorem of Calculus and $F(x) = 0$ for $x < a$, $F(x) = \int_a^b f(x)dx$ for $x > b$. Then $F(\phi(x))' = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x)$, so by the Fundamental Theorem of Calculus, $\int_{-\infty}^{\infty} f(\phi(x))\phi'(x) = \int_{x_a}^{x_b} f(\phi(x))\phi'(x)$ $= \int_{x_a}^{x_b} F(\phi(x))'dx = F(\phi(x_b)) - F(\phi(x_a))$. But by the definitions of x_a and x_b, this is $\int_a^b f(x)dx = \int_{-\infty}^{\infty} f(x)dx$.

c. If we can prove that $f \circ \phi(x)\phi'(x)$ is integrable from $-\infty$ to ∞, the reasoning from part b. will take care of the rest. Here is a quick sketch of the proof. Since ϕ' is continuous on the compact set $[x_a, x_b]$, it is bounded by some constant C. Hence on a segment $[x, y]$, $\frac{\phi(y)-\phi(x)}{y-x}$ is bounded by C as well by the Intermediate Value Theorem. So on a segment $[x, y]$, $\phi(y) - \phi(x) < C(y - x)$. A proof follows by considering partitions of the compact support of f and turning them into partitions of the compact support of $f \circ \phi$.

For another approach, look at pages 66-72 in Spivak.

2) One mistake that was often made on this problem (especially part b.) was assuming that because $g(x) \to h(x)\ \forall x$, $\int g(x) \to \int h(x)$. This would be true if $g(x) \to h(x)$ uniformly over the interval of integration. However, in the cases in question, it often happens that $f(x) \to 0$ $\forall x \neq 0$. But $[-1, 1] - \{0\}$ is not compact, and as it turns out, the convergence is not uniform.

a. $f(x)/x$ is clearly continuous everywhere except possibly at 0. But if $f(0) = 0$, then since f is continuously differentiable, $\lim_{x\to 0} f(x)/x = \lim_{x\to 0} f'(x)/1 = f'(0)$ by L'Hopital. So $f(x)/x$ is bounded on $[-1, 1]$ and continuous everywhere but at one point, hence $f(x)/x$ is integrable.

$\lim_{\epsilon\to 0} I_\epsilon(f) = \lim_{\epsilon\to 0}\int_{-1}^{1}\frac{f(x)x}{x^2+\epsilon^2}dx - i\epsilon\int_{-1}^{1}\frac{f(x)}{x^2+\epsilon^2}dx$. Now

$$|\int_{-1}^{1}\frac{f(x)x}{x^2+\epsilon^2}dx - \int_{-1}^{1} f(x)/x dx|$$

$$= |\int_{-1}^{1}\frac{f(x)\epsilon^2}{x(x^2+\epsilon^2)}dx|$$

$$< |\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}}\frac{f(x)\epsilon^2}{x\epsilon^2}dx| + |\int_{\sqrt{\epsilon}}^{1}\frac{f(x)\epsilon^2}{x(\epsilon+\epsilon^2)}dx| + |\int_{-1}^{-\sqrt{\epsilon}}\frac{f(x)\epsilon^2}{x(\epsilon+\epsilon^2)}dx|$$

Since $f(x)/x$ is bounded on the compact interval $[-1,1]$, each of these integrals goes to 0 as $\epsilon\to 0$, and hence this term is equal to $\int_{-1}^{1}\frac{f(x)}{x}dx$.

We merely need to show that the other term, $\int_{-1}^{1}\frac{f(x)\epsilon}{x^2+\epsilon^2}dx$ disappears. Now we can decompose f into an even part and an odd part, $f(x) = f_e(x) + f_o(x)$. But if we consider the integral of the odd part, it integrates to 0 since the whole integrand is odd. So we need only concern ourselves with the even part: $\int -1^1\frac{f_e(x)\epsilon}{x^2+\epsilon^2}dx$. But $f_e(0)$ and since it is even $f_e'(0)=0$. Hence, by L'Hopital, $\lim_{x\to 0}\frac{f(x)}{x^2} = \lim_{x\to 0}\frac{f'(x)}{2x} = f''(0)/2$. Thus, similarly to before, $\frac{f(x)}{x^2}$ is bounded on $[-1,1]-\{0\}$ by some constant C and since $\frac{f(x)}{x^2+\epsilon^2} < \frac{f(x)}{x^2}$, the integrand is also bounded by C. So the integral is bounded by $2C\epsilon$ and goes to 0 as $\epsilon\to 0$.

This reasoning proves the proposition for $I_-(f)$ as well.

b. Let $g(x) = f(x) - f(0)$, so that $g(0) = 0$. Then

$$I_+(f) = \lim_{\epsilon\to 0}\int_{-1}^{1}\frac{f(0)x}{x^2+\epsilon^2}dx - i\epsilon\int_{-1}^{1}\frac{f(0)}{x^2+\epsilon^2}dx + I_+(g).$$

But the first integrand is odd, and therefore integrates to 0 and we know that $I_+(g)$ exists from part a. so the only term in question is the second integral. We can calculate this one directly: $\int_{-1}^{1}\frac{1}{x^2+\epsilon^2}dx = 1/\epsilon[tan^{-1}(x/\epsilon)]_{-1}^{1} = 1/\epsilon[2tan^{-1}(1/\epsilon)]$. Adding in the terms before the integral, we get $-2if(0)tan^{-1}(1/\epsilon)$ which, as $\epsilon\to 0$, not only converges, but is $-2if(0)(\pi/2) = -i\pi f(0)$. The proof for $I_-(f)$ is analogous in every way.

c. We have essentially done this in parts a. and b. $I_+(f) = -i\pi f(0) + I_+(g)$ and $I_-(f) = i\pi f(0) + I_-(g)$ by part b. But by part a, $I_+(g) = I_-(g) = \int_{-1}^{1} g(x)/x dx$, so $I_+(f) - I_-(f) = -2i\pi f(0)$.

3) a. $\int_I\sum_{i=1}^{n}(-1)^{i-1}\partial P_i/\partial x_i(x_1,...,x_n) = \sum_{i=1}^{n}(-1)^{i-1}\int_I\partial P_i/\partial x_i(x_1,...,x_n)$. By Fubini, this is $\sum_{i=1}^{n}(-1)^{i-1}\int_{I_i}\int_{a_i}^{b_i}\partial P_i/\partial x_i(x_1,...,x_n)dx_i$. Applying

the Fundamental Theorem, we can evaluate the inside integrals to get $\sum_{i=1}^{n}(-1)^{i-1}\int_{I_i} P_i^+(x_1, ...x_{n-1}) - P_i^-(x_1, ...x_{n-1})$.

b. coming soon