Let V be a vector space of dimension $d, \omega \in \tilde{\Omega}_c^d(V)$. For any linear isomrphism $A : \mathbb{R}^d \to V$ we define a function f_A on $\mathbb{R}^d$ by $f_A := A^*(\omega)(e_1, ..., e_d)$ and write

$$\int_V^A \omega := \int_{\mathbb{R}^d} f_A dx_1 ... dx_d$$

.

1. Let $B : \mathbb{R}^d \to V$ be another linear isomorphism. Show that $\int_V^A(\omega) = \pm \int_V^B(\omega)$ where $\pm$ is the sign of $Det(B^{-1}A)$.

Let V be a vector space, $\mathcal{B}$ the set bases in V. We can think about $\mathcal{B}$ as the set of linear isomorphism $B : \mathbb{R}^d \to V$. Given $B', B'' \in \mathcal{B}$ we say that the bases B' and B'' are equivalent $[B' \equiv B'']$ if $Det(B'^{-1}B'') > 0$.

2. Show that $\mathcal{B}$ is a union of two disjoint pieces $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where for any $B', B'' \in \mathcal{B}_i, i = 1, 2$ we have $B' \equiv B''$.

DEFINITION. An orientation $\mathcal{O}$ on V is a choice of one of these two pieces.

DEFINITION. Let $(V, \mathcal{O}_V), (W, \mathcal{O}_W)$ be oriented vector spaces of the same dimension d and $A : V \to W$ an isomorphism of vector spaces. We say that A is compatible with the orientations on V, W if for any [=some] $B \in \mathcal{O}_V$ we have $A \circ B \in \mathcal{O}_W$.

Let V be a 2-dimensional space, e_1, e_2 a basis of V. Since e_1, e_2 are not proportional there exist a well-defined direction of a rotation from e_1 to e_2 in such way that e_1 will reach e_2 without ever crossing the line $\mathbb{R}e_1$ through e_1.

3. Prove that two bases $B' = (e_1', e_2'), B'' = (e_1'', e_2'')$ of V are equivalent if and only if the direction of the rotation from e_1' to e_2' coincides with the direction of the rotation from e_1'' to e_2''.

Let $U \subset V$ be an open set. We denote by $\in \tilde{\Omega}_c^r(U)$ the set of smooth r-forms on U *with compact support.* In other words $\tilde{\Omega}_c^r(U)$ is the set of smooth r-forms ω on U such that there exists a compact set $D \subset U$ such that $\omega(x) = 0$ for $x \in U - D$. For any $\omega \in \tilde{\Omega}_c^r(U)$ we denote by $\tilde{\omega}$ the r-form on V such that $\tilde{\omega}(x) = \omega(x)$ if $x \in U$ and $\tilde{\omega}(x) = 0$ if $x \in V - U$. It is easy to see that $\tilde{\omega} \in \tilde{\Omega}_c^r(V)$. If $\mathcal{O}$ is an orientation of V then for any $\omega \in \tilde{\Omega}_c^d(U)$ we define

$$\int_U^{\mathcal{O}} \omega := \int_V^{\mathcal{O}} \tilde{\omega}$$

Let $(V, \mathcal{O}_V), (W, \mathcal{O}_W)$ be oriented vector spaces of the same dimension $d, U_V \subset V, U_W \subset W$ open subsets,$\phi : U_V \to U_W$ a continuously differentiable map which is a)one-to-one and onto and

b) for any $u \in U_V$ the linear map $D(\phi)(u) : V \to W$ is an isomorphism compatible with the orietations $\mathcal{O}_V, \mathcal{O}_W$.

4. a) Show that for any $\omega \in \tilde{\Omega}^d_c(U_W)$ the form $\phi^*(\omega)$ belongs to $\tilde{\Omega}^d_c(U_V)$.

b)Show that for any $\omega \in \tilde{\Omega}^d_c(U_W)$ we have $\int_{U_W}^{\mathcal{O}_W} \omega = \int_{U_V}^{\mathcal{O}_V} \phi^*(\omega)$

Let $S^n \subset \mathbb{R}^{n+1}$ be the sphere of radius $1, S^n = \{x_0, ..., x_n | x_0^2 + ... + x_n^2 = 1\}$ and $F_n : S^n \to S^n$ be the antipodal involution $F_n(x_0, ..., x_n) = (-x_0, ..., -x_n)$.

5. a)Show how to define an orientation on S^n.

b)Find for which n the map preserves an orientation.

6. Let $f : S^2 \to \mathbb{R}^6$ be the map given by $f(x, y, z) = (x^2, y^2, z^2, xy, xz, yz)$.

a) Show that the image of Σ of f is a smooth surface in $\mathbb{R}^6$.

b)Show that Σ is not orientable.