1. Let $C = B^{-1}A$.

$$\begin{aligned} f_B(C(x))\det(C) &= \det CB^*(\omega)(C(x)(e_1...e_d)) \\ &= \det C\omega(BB^{-1}A(x)(Be_1...Be_d)) \\ &= \det(BC)\omega \circ A(x)(Ae_1...Ae_d) = f_A(x) \end{aligned}$$

Therefore,

$$\begin{aligned} \int_V^A(\omega) &= \int_{\mathbb{R}^d} f_A(x)dx_1...dx_d \\ &= \int_{\mathbb{R}^d} f_B(C(x))\det(C)dx_1...dx_d \\ &= sign(\det(C))\int_{\mathbb{R}^d} f_B(C(x))|\det(C)|dx_1...dx_d \\ &= sign(\det(C))\int_{\mathbb{R}^d} f_B(x)dx_1...dx_d = sign(\det(C))\int_V^B(\omega) \end{aligned}$$

2. PLEASE NOTE THAT YOU CANNOT REFER TO B_1 AND B_2 WITHOUT DEFINING THEM!!!!

To show that this relation actually partitions B into some number of sets we must show that it is an equivalence relation. That is to say that $B' \equiv B'$, $B' \equiv B'' \Rightarrow B'' \equiv B'$ and $B' \equiv B'' \equiv B''' \Rightarrow B' \equiv B'''$. All of these facts follow from the fact that the determinant is multiplicative. Now to show that this relation partitions B into exactly two sets. Firstly there must be more than one because if we have some $B' \in B$ then $-Id \circ B'$ is not equivalent to B'. So now to show that there are exactly two equivalence classes we need only show that if B'' and B''' are not equivalent to B' then they are equivalent to each other. But $\det({B''}^{-1}B''') = \det({B''}^{-1}B')\det({B'}^{-1}B''')$. So if the latter two are negative the left hand side must be positive just as we had hoped. Notice that I could at any point in there actually define B_1 as the set of things equivalent to B' and B_2 as the set of things not equivalent to B' and then the last sentence would be showing that B_2 is actually an equivalence class.

3. You pretty much all got this so I don't want to bother with it. Basically bother with trig identities and find out the determinant in question is actually the sine of the relevant angle.

4.a) You all did pretty well on showing that the set $Im(\phi^{-1}(U_w))$ should be the set where our new guy is non-zero. Unfortunately none of you bothered to show it was compact. So i'll just do that part. By the inverse function theorem ϕ^{-1} has continuous derivative whenever $D\phi$ is an isomorphism. But by the assumption (note none of you mentioned this assumption so your solution must be wrong) this is always the case. Therefore ϕ^{-1} is continuous, and the continuous image of a compact set is compact.

b) I'M GOING TO SCREAM IF ANOTHER PERSON PRETENDS THAT $\phi \circ O_V$ IS LINEAR!!!

Chose A and B bases of V and W respectively with the proper orientations.

Lemma: $f_A(x) = f_B(B^{-1} \circ \phi^{-1} \circ A(x))\det(B^{-1} \circ D\phi^{-1}(A(x)) \circ A)$

$$\begin{aligned} f_A(x) &= \omega(A(x))(Ae_1...Ae_d) = \det(A)\omega(A(x)(e_1...e_d)) \\ &= \det(B^{-1} \circ D\phi^-1(A(x)) \circ A \circ D\phi(A(x)) \circ B)\omega(A(x)(e_1...e_d) \\ &= \det(B^{-1} \circ D\phi^-1(A(x)) \circ A)B^*(\phi^*(\omega))(B^{-1}\phi^{-1}A(x)(e_1...e_d)) \\ &= f_B(B^{-1} \circ \phi^{-1} \circ A(x)) \det(B^{-1} \circ D\phi^{-1}(A(x)) \circ A) \end{aligned}$$

Now we just integrate both sides using the change of variables formula just as in 1 and get the desired result.

5.a) The fact that a manifold which is the boundary of a compact set is orientable was done in class.

WHAT ALL OF YOU DID IS WRONG! LOOK BACK AT WHAT YOU DID!!

If what you did worked then all manifolds would be orientable (you used nothing about S^n). The specific reason which it doesn't work is that your definition is dependent on your choice of parameterization. Although the tangent space is not dependent on parameterization the actual matrix DF most certainly is (notice the difference here... the image of the matrix doesn't depend but the matrix itself does). Therefore your definition doesn't make sense without specifying which parameterizations you take and furthermore it might not be consistent everywhere because the same parameterization doesn't work everywhere.

b) Choose any $\sigma \in S^n$, then $\exists F : U \to S^n$ a parameterization of S^n at σ. To see if F_n preserves orientation, we see if the tangent planes at σ and $F_n(\sigma)$ are oriented the same way. So consider $\det(DF(0)^{-1} \circ D(F_n(F(0)))) = \det(DF_n(\sigma)$ by applying the chain rule. But, $DF_n(\sigma) = -Id_{n+1}$. Therefore, $\det(DF_n(\sigma))$ is positive iff n is odd. So the map preserves orientation iff n is odd.

6.a) I really don't feel like going through the details of all this, basically there are two methods and one pitfall for each of them:

method 1: Use some theorem which we've apparently proved in class that if you have a function that is smooth and whose derivative always has the rank of the dimension of your manifold then the image is also a smooth manifold (this follows pretty quickly from the parameter definition of a manifold).

pitfall 1: not stating the theorem... using words (usually embedding) incorrectly.

method 2: directly showing this using the implicit function definition.

pitfall 2: not dealing with the square roots properly.

method 1 is much quicker and easier.

b) okay... here's the big problem: WHY MUST THE ORIENTATION ON Σ COME FROM ONE ON S^2???? Just because that orientation doesn't work needn't mean that Σ couldn't have some other orientation, right? Anyway, here's a proof.

Assume we have some orientation on Σ, then we can pull this orientation back to an orientation on S^2 (this is opposite from what you all did) by definining a basis on the tangent space to be positively oriented if its image in the tangent space of the image point on Σ is positively oriented. This clearly gives an orientation on S^2. Now we have shown that F_2 is orientation reversing (we only showed that it reversed one particular orientation but it is true that the "orientation preserving"-ness of a map is independent of choice of orientation, i'm not sure if we've proved this in class but i'm certainly not going to try and prove this here). But, F_2 clearly does not change the orientation pulled back from Σ (since antipodes came from the same point on Σ). This is a contradiction, therefore, Σ is non-orientable.