# MATH 23B SOLUTION SET 8

1) We need to calculate $f(w_1, w_2) = \phi*(\omega)(w_1, w_2)(1, 0)$ and $g(w_1, w_2) = \phi*(\omega)(w_1, w_2)(0, 1)$. $\phi*(\omega)(w_1, w_2)(1, 0) = \omega_{\phi(w_1,w_2)}(D\phi(1, 0))$ where $D\phi$ is taken at $(w_1, w_2)$. So $D\phi(1, 0) = (\frac{\partial \phi_1}{\partial w_1}, \frac{\partial \phi_1}{\partial w_2}, \frac{\partial \phi_1}{\partial w_3})$ if $\phi(w) = (\phi_1(w_1, w_2), \phi_2(w_1, w_2), \phi_3(w_1, w_2)) \in V$. Calculating this yields $D\phi(1, 0) = (2w_1, w_2, 0)$. Similarly, $D\phi(0, 1) = (0, w_1, 2w_2)$. Now at the point in V, $\phi(w_1, w_2) = (w_1^2, w_1w_2, w_2^2)$, $\omega$ is $w_1^2e^1 + w_1w_2e^2 + w_2^2e^3$, so applying this to the vectors calculated previously we get $f(w) = 2w_1^3 + w_1w_2^2$ and $g(w) = w_1^2w_2 + 2w_2^3$.

2) For any point $x = (x_1, x_2)$ and vector $v = (v_1, v_2)$, consider the following: $v_1 = e^1(v) = \phi*(\omega_1)_x(v) = \omega_1(\phi(x))(D\phi(v)) = e^1(D\phi(v)) = \frac{\partial \phi_1}{\partial x_1}v_1 + \frac{\partial \phi_1}{\partial x_2}v_2$. So $\frac{\partial \phi_1}{\partial x_1}v_1 + \frac{\partial \phi_1}{\partial x_2}v_2 = v_1$ for all $(v_1, v_2) \in \mathbb{R}^2$. Thus we must have $\frac{\partial \phi_1}{\partial x_1} = 1$ and $\frac{\partial \phi_1}{\partial x_2} = 0$. We can do a similar analysis with $\omega_2$ and get $\frac{\partial \phi_2}{\partial x_1} = 0$ and $\frac{\partial \phi_2}{\partial x_2} = 1$. These formulas apply at all $(x_1, x_2)$ and together imply that $\phi_1 = x_1 + C_1$, $\phi_2 = x_2 + C_2$ for some constants $C_1$ and $C_2$. (If you'd like to prove this, take a line integral from $(0, 0)$ to $(x_1, x_2)$ which we can do for sure since $\phi$ is smooth.)

3) The function in question from the fall is $f(x) = 0$ for $x \leq 0$, $f(x) = e^{-1/x}$ for $x > 0$. This is smooth at 0 and goes to 1 as $x \to +\infty$. Here is a sketch for how to produce the answer to part b: Look at $Cf(x+1)f(-x-1)$ for some constant $C$ and set $C$ to the value required by the integral. Part a can be determined by integrating part b.

4) a. We first show that the first derivative exists and is continuous. Calculate $\frac{F^\epsilon(y+h)-F^\epsilon(y)}{h}$

$$= \frac{1}{h}(\int_{\mathbb{R}} f_\epsilon(x)F(y + h - x)dx - \int_{\mathbb{R}} f_\epsilon(x)F(y - x)dx)$$

$$= \frac{1}{h}(\int_{\mathbb{R}} f_\epsilon(x' + h)F(y - x')dx' - \int_{\mathbb{R}} f_\epsilon(x)F(y - x)dx)$$

$$= \frac{1}{h}(\int_{\mathbb{R}} F(y - x)(f_\epsilon(x + h) - f_\epsilon(x))dx.$$

Since the integral is of a bounded sequence of functions over a compact set, we can take the limit inside the integral and as we take $h \to 0$ we get

$$\int_{\mathbb{R}} F(y - x)f'_\epsilon(x)dx.$$

Clearly we can perform a similar operation repeatedly to show that all derivatives exist and are continuous.

b. The integral $F^\epsilon(y)$ is bounded above and below by the maximum and minimum values respectively of $F$ on the interval $[y-\epsilon, y+\epsilon]$ since outside this range $f_\epsilon(x)$ and hence the integrand are zero. Since $F$ is continuous, for every $\alpha > 0$ and every $y \in \mathbb{R}$ there is an $\epsilon$ s.t. this min and max differ by $\alpha$. However, to prove *uniform* convergence, we need to show that this $\epsilon$ is independent of $y$. We need $F$ to be *uniformly* continuous. Luckily, we now know that $F$ has compact support, so it must be uniformly continuous. (To see this, note that $max_{[y-\epsilon,y+\epsilon]}F - min_{[y-\epsilon,y+\epsilon]}F$ is a continuous function on a compact set, and hence is bounded.)

c. This follows directly from the observation above, that the integrand is only nonzero in an $\epsilon$-neighborhood of the center. In this situation, no points in $C$ are in that region, so at no point is the integrand nonzero.

d. There are several ways to do this. The most basic makes the $f_\epsilon(x)$ a radial function s.t. it is nonzero only inside a hypersphere of radius $\epsilon$ and integrates to 1. In this case, the proofs are completely analagous, as they are in the similar construction with a hypercube instead.

5) a. There are several ways to calculate this. The simplest is to look at

$$\begin{aligned} I_n^2 &= (\int_{\mathbb{R}} e^{-nx^2} dx)(\int_{\mathbb{R}} e{-}ny^2 dy) \\ &= \int_{\mathbb{R}^2} e^{-nx^2-ny^2} dxdy \\ &= \int_{\mathbb{R}^2} e^{-nr^2} r d\theta dr \\ &= 2\pi(-1/2n)[e^{-nr^2}]_0^\infty \\ &= \pi/n. \end{aligned}$$

The following is wrong. I'll come back to part b as soon as I can:

b. Outside of any $\epsilon$-neighborhood of 0, $f$ is bounded above less than 1. Hence, for large enough $n$, we can ignore the parts of the integral outside of $[-\epsilon, \epsilon]$ for any $\epsilon$. (The integral of those parts goes to zero as $r^n$ for some $r < 1$.) Now we approximate $f(x)^n$ near 0 by using the Taylor expansion of $ln|f(x)|$, which is $ln(f(0)) + f'(0)/f(0)x + ((f''(0)f(0) - f'(0)^2)/f(0)^2)x^2$. Now since $f(0) = 1$ is a local maximum, $f'(0) = 0$ and this reduces to $f''(0)x^2$. So $f(x)^n$ can be approximated by $e^{nf''(0)x^2}$.