Let Σ be a closed subset of $\mathbb{R}^n$. We say that Σ is a d-dimensional *submanifold with a boundary* if there exists a closed subset $\partial\Sigma \subset \Sigma$ [called the bounday of Σ] such that

a) For any point $\sigma \in \Sigma - \partial\Sigma$ there exist an open neighbourhood U_σ of σ in $\mathbb{R}^n$, an open neighbourhood U^0_σ of 0 in $\mathbb{R}^d$ and a smooth map $\phi_\sigma : U^0_\sigma \to U \cap \Sigma$ which is one-to-one, onto, $\phi_\sigma(0) = \sigma$ and $D\phi(0) : \mathbb{R}^d \to \mathbb{R}^n$ is an imbedding. [In other words any point σ of $\Sigma - \partial\Sigma$ is smooth].

b) For any point $\sigma \in \partial\Sigma$ there exist an open neighbourhood U_σ of σ in $\mathbb{R}^n$, an open neighbourhood U^0_σ of 0 in $\mathbb{R}^d$ and a smooth map $\phi_\sigma : U^0_\sigma \to \mathbb{R}^n$ such that $\phi_\sigma(0) = \sigma$ and $D\phi(0) : \mathbb{R}^d \to \mathbb{R}^n$ is an imbedding and the restriction of ϕ_σ on $U^0_\sigma(+)$ is a map from $U^0_\sigma(+)$ to $U \cap \Sigma$ which is one-to-on and onto where $U^0_\sigma(+) = \{(x_1, ...x_d) \in U^0_\sigma | x_d \geq 0\}$.

We call such a map ϕ_σ *a coordinate chart* for Σ at σ.

1.a) Show that $\partial\Sigma$ is a $(d-1)$-dimensional submanifold of $\mathbb{R}^n$.

b)Let $\mathcal{O}_\Sigma$ be an orientation on Σ. Show how to define an orienation $\mathcal{O}_{\partial\Sigma}$ on $\partial\Sigma$. Consider in details the case when $d = 1$ and $d = 2$.

Let $S^2_+ = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1, z > 0\}, D = \{(a, b) \in \mathbb{R}^2 | a^2+b^2 < 1\}$ and $\phi : D \to S^2_+$ be given by $\phi(a, b) := (a, b, \sqrt{1 - (a^2 + b^2)})$.

2.a)Show that for any $(a, b, c) \in S^2_+$ and any choice of a vector $v_3 \in \mathbb{R}^3$ at (a, b, c) which "looks outside" v_1, v_2, v_3 is a basis in $\mathbb{R}^3$ where $v_1 := D\phi(a, b)(l_1), v_2 := D\phi(a, b)(l_2)$ where l_1, l_2 is the standard basis in $\mathbb{R}^2$.

b)Show that the orientation of this basis in $\mathbb{R}^3$ is "positive" [:=coincides with the standard orientation of $\mathbb{R}^3$].

Let Σ be a smooth d-dimensional submanifold of $\mathbb{R}^n, \sigma_1, ...\sigma_N$ points of Σ and $\phi_{\sigma_i} : U^0_i \to U_i$ coordinate charts such that $\bigcup_{i=1}^N U_i = \Sigma$.

3.a)Using this system of coordinate charts define a notion of a r-differential form on Σ. [Look in details at the case $r = 0$].

Assume that Σ is orientable and we have chosen an orientation $\mathcal{O}$ on Σ.

b)Explain how to for any differential d-form ω on Σ with compact support define the integral $\int^{\mathcal{O}}_\Sigma \omega$ of ω over Σ.

Let $U \subset \mathbb{R}^n$ be an open set and $F : U \to \mathbb{R}$ a smooth function. Then for any $u \in U$ we have a linear map $DF(u) : \mathbb{R}^n \to \mathbb{R}$. In other words we can consider $DF(u)$ as an element in $V^\vee = \Omega^1(V)$. So DF is a function from U to $\Omega^1(V)$. In other words DF is a differential form.

4. Let Σ be smooth d-dimensional submanifold of $\mathbb{R}^n, F : \Sigma \to \mathbb{R}$ a smooth function. Define the 1-form DF.

Let Σ be an oriented 1-dimensional manifold with the ends at point a and b and the orientation goes "from a to b.

5. Show that for any smooth function $F : \Sigma \to \mathbb{R}$ we have $\int_{\Sigma}^{\mathcal{O}} DF = F(b) - F(a)$