1.a) Consider $\tilde{U}_\sigma^0$ the set $(x_1, ..., x_d) \in U_\sigma^0 | x_d = 0$ and $\tilde{\phi}_\sigma$ a map $\tilde{U}_\sigma^0 \to \mathbb{R}^n$ the restriction of ϕ_σ. I claim the image of this map is exactly $\partial\sigma \cap U_\sigma$.

Before that, I want to show that $\tilde{\phi}_s igma$ sends open sets to open sets and closed sets to closed sets and that the inverse image of open sets are open and the inverse image of closed sets are closed ... THIS DOES NOT COME FOR FREE FROM CONTINUITY AS MANY OF YOU SEEMED TO THINK! The only thing you get for free is that the inverse image of open sets is open.

image of open sets: By the inverse function theorem there exists an inverse function which is continuous. Now the inverse image of open sets under the inverse map must be open. Hence the forward image of open sets must be open.

forward and backwards images of closed sets: Take compliments and use the fact that the function is one-to-one and onto.

Anyway... back to the question at hand. A couple definitions i want to make first:

an interior point: a point which has an open neighborhood still contained in the set.

a boundary point: a point that is a limit point of both the set and its compliment. That is to say that any open neighborhood of these points intersect both the set and its compliment.

fact: Any point in a set is either an interior point or a boundary point. (this is very easy, i'll leave the proof to you.)

(note, however, points outside the set can still be boundary points, a set being closed is the same thing as saying it includes its boundary and a set being open is the same as it being disjoint from its boundary)

My claim is clearly follows from the fact that boundary points map to boundary points and interior points map to interior points. (Since the boundary of $U_\sigma^0(+)$ is $\tilde{U}_\sigma^0$ and the boundary of σ is $\partial\sigma$... i'll leave these two facts as an exercise to the reader).

MANY OF YOU SAID STUFF LIKE THE "CLOSED PART" OF ONE MUST GO TO THE "ONLY CLOSED PART" OF THE OTHER... firstly you never defined those terms, secondly you never proved that statement.

Now... That interior points map to interior points follows from the fact that the forward image of an open set is open. The inverse image of an interior point must also be interior since the inverse image of an open set is open, hence the forward image of a boundary point must be a boundary point.

Therefore we now have an induced map $\tilde{U}_\sigma^0 \to \partial\sigma \cap U_\sigma$. A simple check shows that this gives us a parameterization at σ and hence the boundary is a $d-1$ dimensional manifold.

(Obviously this proof can be shortened by not defining anything, but i think that these definitions help you see the picture better)

b) See set 7 for why most of yours answer doesn't work.

The correct method is of course to say a basis is positively oriented if when you take that basis together with an "outward looking" vector it is positive. This was done in class and i believe in the long thing Prof. Kazhdan put on the website, so i won't repeat it here.

in the cases of $d = 1$ or 2 you were expected to say that if you have a one dimensional manifold with boundary it looks like a curve and the orientation is given by a direction from one endpoint to the other, so the boundary orientation is a positive sign on the ending point and a negative sign on the beginning point. For two dimensions we basically have a sheet and the orientation on the outside part of the boundary is given by the direction of rotation from the second basis vector to the first and on any holes it is given in the opposite direction. (i may have accidentally switched those... check it yourself).

2.a) This is essentially a pretty easy question, many of you did way too much work on it or missed the point. Clearly v_1 and v_2 form a basis for the tangent space (since the derivative is an embedding). Since v_3 looks outwards it is not in the tangent space and so it is not in the span of the other two vector, so it is linearly independent from them. Therefore they form a basis.

b) The difficult part is finding a good definition of "looking out". Here's the best one. $v_3 = av_1 + bv_2 + c(v_1 \times v_2)$ where $c > 0$. Now, to check the orientation of (v_1, v_2, v_3) we need to take the determinant of the matrix $[v_1\ v_2\ v_3]$ where these vectors are written as columns. Now, $\det[v_1\ v_2\ av_1 + bv_2 + c(v_1 \times v_2)] = c\det[v_1\ v_2\ v_1 \times v_2] > 0$ (the last step is just a simple direct computation).

3,4 These should be done on Kazhdan's big stoke's thing on the web... let me know if you need more info on this... the main problems were:

i) using things before you defined what they were

ii) not showing that your definitions were independent of choice of your parameterization (see the whole orientation thing from set 7 to see how this can screw up definitions) and independent of your choice of partition of unity.

5. This is just a straightforward application of the fundemental theorem of caculus which basically all of you got.