Let $\Sigma', \Sigma''$ be two smooth submanifolds in $\mathbb{R}^{n'}$ and $\mathbb{R}^{n''}$ and $f : \Sigma' \to \Sigma''$ a map.

1. a)What does it mean that $f$ is smooth.

b)Explain how to define the "natural" map $f^* : \tilde{\Omega}^r(\Sigma') \to \tilde{\Omega}^r(\Sigma')$ for any smooth map $f : \Sigma' \to \Sigma''$.

Let $\Sigma \subset \mathbb{R}^n$ be a smooth $d$-dimensional manifold.

2.a) Show that for any $\omega' \in \tilde{\Omega}^{r'}(\Sigma), \omega'' \in \tilde{\Omega}^{r''}(\Sigma)$ there exists a smooth $r'+r''$-form $\omega \in \tilde{\Omega}^r(\Sigma), r = r'+r''$ such that for any $\sigma \in \Sigma$ we have $\omega'(\sigma) \wedge \omega''(\sigma) = \omega(\sigma)$.

We denote the form $\omega \in \tilde{\Omega}^r(\Sigma)$ by $\omega' \wedge \omega''$.

b)Show that $\omega' \wedge \omega'' = (-1)^{r'r''}\omega'' \wedge \omega'$.

Let $V$ be a vector space of dimension $n$ and $e_1, ..., e_n$ a basis of $V$ and $e^1, ..., e^n$ be the dual basis. We denote by $x_i, 1 \leq i \leq n$ linear functions on $V$ given by $x_i(v) := e_i(v)$.

As in the problem set 2 we denote by $\mathcal{A}(r,n)$ be the set of subsets $I$ of $[1, ..., n]$ such that $|I| = r$. For any $I = \{i_1 < ... < i_r\} \in \mathcal{A}(r,n)$ we associate an $r$-linear form $\omega^I$ on $V$ by

$$\omega^I(v_1, ..., v_r) := \sum_{\sigma \in S_r} sign(\sigma) \prod_{1 \leq k \leq r} e^{i_{\sigma(k)}}(v_k)$$

.

3. Show that for any $I = \{i_1 < ... < i_r\} \in \mathcal{A}(r,n), v \in V$ we have $dx_{i_1} \wedge ... \wedge dx_{i_r}(v) = \omega^I$

As you know [see the problem set 2] the set $\omega^I, I \in \mathcal{A}(r,n)$ is a basis of the space $\Omega^r(V)$. Therefore for any open set $U \subset \mathbb{R}^n$ and any differensial form $\omega \in \tilde{\Omega}^r(U)$ we can write $\omega$ as a sum

$$\omega = \sum_{I=\{i_1<...<i_r\} \in \mathcal{A}(r,n)} f_I dx_{i_1} \wedge ... \wedge dx_{i_r}$$

where $f_I, I \in \mathcal{A}(r,n)$ are smooth functions on $U$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, $a \in \mathbb{R}, \Sigma_a := f^{-1}(a) \subset \mathbb{R}^n$. Assume that for any $\sigma \in \Sigma_a$ the map $d(f)(\sigma) : \mathbb{R}^n \to \mathbb{R}$ is onto. [From now on I'll write $df$ instead of $Df$]. Then, as you know, $\Sigma_a$ is a smooth submanifold of $\mathbb{R}^n$ of dimension $n-1$. Let $\eta$ be the $n$-form on $\mathbb{R}^n$.

4. a)Show that for any $\sigma \in \Sigma_a$ there exists an open neighbourhood $U$ of $\sigma$ in $\mathbb{R}^n$ and $n-1$-from $\tilde{\omega}$ on $U$ such that the restriction of $\eta$ on $U$ is equal to the product $\tilde{\omega} \wedge df$. [In other words, for any $u \in U$ we have $\tilde{\omega}(u) \wedge df(u) = \eta(u)$].

b)Show that the restriction $\omega_a(\eta)$ of the form $\tilde{\omega}$ on $\Sigma_a \cap U$ does not depend on a choice of the form $\tilde{\omega}$. We denote this $n-1$-from on $\Sigma_a \cap U$ by $\omega_a(\eta)|U$.

As follows from b), for any point $\sigma \in \Sigma_a$ we have defined an element $\omega_a(\eta)(\sigma)$ in $\Omega^{n-1}(T_{\Sigma_a}(\sigma))$.

c)Show the map $\sigma \to \omega_a(\eta)(\sigma)$ is a smooth $n-1$-form on $\Sigma_a$. We denote it by $\omega_a(\eta)$.

d)Show that for any $a \in \mathbb{R}$ the $n-1$-form $\omega_a(dx^1 \wedge ... \wedge dx^n)$ defines an orientation on $\Sigma_a$. We denote this orintation as $\mathcal{O}_a$.

e)Let $\eta$ be an $n$-form on $\mathbb{R}^n$ with compact support,$\mathcal{O}$ be the standard orientation on $\mathbb{R}^n$ [the one corresponding to the form $dx^1 \wedge ... \wedge dx^n$ ]. Show that $\int_{\mathbb{R}^n} \eta = \int_{\mathbb{R}}^{\mathcal{O}} f(a)da$ where $f(a) := \int_{\Sigma_a}^{\mathcal{O}_a} \omega_a(\eta)$.

f)Compute the form $\omega := \omega_1(\eta)$ in the case when $n = 3, \eta(v) := e^1 \wedge e^2 \wedge e^3$ and $f(x,y,z) = x^2 + y^2 + z^2$. [Choose some system of local coordinates on $\Sigma_1 = S^2$.

g)Show that for any rotation $f : S^2 \to S^2$ we have $f^*(\omega_1) = \omega_1$.

h)Choose an orientation on $S^2$ and compute $\int_{S^2}^{\mathcal{O}} \omega$.

5. Do the problems 6.5.5; 6.5.13; 6.6.2; 6.9.3 and 6.9.7