MATH 23B SOLUTION SET 10

1) a. We don't know what differentiable means on manifolds, they don't have the structure to define it, in general. In this case, out manifolds are embedded in $\mathbb{R}^n$, so we could use the structure of the embedding, but it is still nicer to use charts. In this case, we do the usual trick of taking everything back to $\mathbb{R}^n$ through the charts: if $\phi_1 : U_1 \to V_1$ with U and V opens subsets of $\mathbb{R}^n$ and Σ' respectively and $\phi_2 : U_2 \to V_2$ is a similar map to Σ'', we can define smoothness for the part of f that maps anything in V_1 to anything in V_2 by saying it is equivalent to $\phi_2^{-1} \circ f \circ \phi_1 : U_1 \to U_2$ being smooth. Then we say that f is smooth if this is true for all charts on Σ' and Σ''.

b. The "natural" map here is simply the pullback: $f^*(\omega)_p(v_1, ..., v_r) = \omega_{f(p)}(Df \circ v_1, ..., Df \circ v_r)$. Here we have defined Df in a similar way to the definition of smoothness in part a. Using pairs of charts, we turn f into a set of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. The derivative is defined as the derivative of these functions.

2) a. We know that there is a particular $r' + r''$-form at every point $\sigma \in \Sigma$ defined by $\omega(\sigma) = \omega'(\sigma) \wedge \omega''(\sigma)$. The only possible problem would be on the intersection of charts. However, because we know that ω'' and ω' are consistently defined from chart to chart, ω is as well (the value of ω when applied to a set a vectors is explicitly defined by the values of ω'' and ω'.). So we need only consider the question of smoothness in a particular chart. With respect to a chart, we can write

$$\omega'(p) = \sum_I f_I'(p)dx_{i1} \wedge ... \wedge dx_{ir'}$$

$$\omega''(p) = \sum_I f_I''(p)dx_{i1} \wedge ... \wedge dx_{ir''}$$

where f' and f' are smooth functions and the sums are over all subsets I of $\{1, ..., n\}$ with r' or r'' elements. Then we can write ω in a similar form which establishes smoothness: $\omega(p) = \frac{1}{r'!r''!} \sum_{(I',I'')} f_{I'}'(p) f_{I''}''(p) \times$ (linear combination of dx_i).

b. We know that this equation is true for antilinear forms on vector spaces (midterm), so it is true at every point by the definition of ω.

3) $dx_{i1} \wedge ... \wedge dx_{ir} = \sum_\sigma sign(\sigma)dx_{i\sigma(1)} \otimes ... \otimes dx_{i\sigma(r)}$ since this is the only completely antilinear form in the variables $\{x_{i1}, ..., x_{ir}\}$ which sends the set of vectors $\{e_{i1}, ..., e_{ir}\}$ to 1 and anything with any other

basis vectors to zero. But because $dx_j := e^j$, this is simply what we wanted to prove.

4) a. Write
$$\eta(x) = k(x)dx_1 \wedge ... \wedge dx_n$$
$$\tilde{\omega}(x) = \sum_i g_i(x)dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ... \wedge dx_n$$
$$df(x) = \sum_i \frac{\partial f}{\partial x_i}(x)dx_i.$$

Then
$$\tilde{\omega} \wedge df(x) = \sum_i g_i(x)\frac{\partial f}{\partial x_i}(x)(-1)^{n-i}dx_1 \wedge ... \wedge dx_n.$$

So we can have a solution if there exists a set of smooth functions $g_i(x)$ s.t. $k(x) = (-1)^{n-i}g_i(x)\frac{\partial f}{\partial x_i}(x)$. As it turns out, this can be done, because df is onto, so it is never 0 and the functions $f'_i := \frac{\partial f}{\partial x_i}(x)$ are never all zero. Hence at x, some particular f'_i is nonzero and we can choose a neighborhood U around x so that f'_i remains nonzero on this neighborhood and df remains onto. Then we can define $g_i(x) = (-1)^{n-i}k(x)/f'_i(x)$ and make all the other g_j zero.

As it turns out, this can be done globally, as well, if df is globally onto, using a partition of unity: If we consider the sets $O_i = \{x \in \mathbb{R}^n | f'_i(x) \neq 0\}$, these are all open ($f'^{-1}_i(0)$ is closed) and cover $\mathbb{R}^n$. Hence we can find a partition of unity $h_i(x)$ subordinate to these open sets and define $(-1)^{n-i}g_i(x) = h_i(x)/f'_i(x)$ on O_i, 0 otherwise. You can check the definition of a partition of unity subordinate to an open cover to see that this is indeed defined and smooth.

b. The set of all forms $\tilde{\omega}$ sthat satisfy the conditions of part a is $\{\tilde{\omega}_0 + \nu | \nu \wedge df = 0\}$ where $\tilde{\omega}_0$ is a particular solution like the one found in part a.

to be continued... in review section.