MATH 23B FIRST MIDTERM

due March 3

Let V be a vector space. For any $\omega' \in \Omega^{r'}(V), \omega'' \in \Omega^{r''}(V)$ define a product $\omega' \wedge \omega'' \in \Omega^r(V), r = r' + r''$ by the formula $\omega' \wedge \omega'' = \frac{(r'+r'')!}{r'!r''!} Alt(\omega' \otimes \omega'')$ where $\omega' \otimes \omega'' \in \mathcal{B}^r(V)$ is given by $\omega' \otimes \omega''(v_1, ..., v_r) := \omega'(v_1, ..., v_{r'})\omega''(v_{r'+1}, ..., v_{r'+r''})$. [Remember that $r = r' + r''$].

1. Show that

a) $\omega' \wedge \omega'' = (-1)^{r'r''}\omega'' \wedge \omega'$

b)The product is distributive [that is show that $(\omega'_1 + \omega'_2) \wedge \omega'' = \omega'_1 \wedge \omega'' + \omega'_2 \wedge \omega''$.]

c)For any linear operator $A : V \to V$ we have

$\Omega^r(A)(\omega' \wedge \omega'') = \Omega^r(A)(\omega') \wedge \Omega^r(A)(\omega'')$.

d)For any three forms $\omega_i \in \Omega^{r_i}(V), i = 1, 2, 3$ we have

$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \frac{(r_1+r_2+r_3)!}{r_1!r_2!r_3!} Alt(\omega_1 \otimes \omega_2 \otimes \omega_3)$

and

$\omega_1 \wedge (\omega_2 \wedge \omega_3) = \frac{(r_1+r_2+r_3)!}{r_1!r_2!r_3!} Alt(\omega_1 \otimes \omega_2 \otimes \omega_3)$

Remark. The part d) shows that the product $(\omega', \omega'') \mapsto \omega' \wedge \omega''$ is associative.

Let $d = dimV$. We fix a nonzero element $\eta \in \Omega^d(V)$. For any $\omega \in \Omega^{d-1}(V)$ and $\lambda \in V^\vee = \Omega^1(V)$ the product $\lambda \wedge \omega$ belongs to $\Omega^d(V)$. Since $dim(\Omega^d(V)) = 1$ we have $\lambda \wedge \omega = c\eta$ where $c = c(\lambda, \omega) \in \mathbb{R}$. Fix $\omega \in \Omega^{d-1}(V)$ and consider the map $f_\omega : V^\vee \to \mathbb{R}$ given by $f_\omega(\lambda) := c(\lambda, \omega)$.

2. a)Show that the function f_ω is linear.

Since the function f_ω is linear we can consider f_ω as an element of the space $(V^\vee)^\vee$ dual to the space $V^\vee$. Since the space V is finite-dimensional we can identify canonically the space $(V^\vee)^\vee$ with V. Therefore we have constructed a map $B_\eta : \Omega^{d-1}(V) \to V$.

b)Prove that the map $B_\eta : \Omega^{d-1}(V) \to V$ is linear.

c)Show that for any non-zero $a \in \mathbb{R}$ we have $B_{a\eta} = a^{-1}B_\eta$

Let $e_i, 1 \leq i \leq d$ be a basis of V, e^i the dual basis of $V^\vee$. For any $i, 1 \leq i \leq d$ let $\omega_i := e^1 \wedge ....e^{i-1} \wedge e^{i+1} \wedge .... \wedge e^d \in \Omega^{d-1}(V)$.

d)Show that $\omega_i, 1 \leq i \leq d$ is a basis of the space $\Omega^{d-1}(V)$ and find $B_\eta(\omega_i), 1 \leq i \leq d$ for $\eta := e^1 \wedge .... \wedge e^d$.

e)Prove that the map $B_\eta : \Omega^{d-1}(V) \to V$ is an isomorphism.

For any linear operator $A : V \to V$ we define $\tilde{A} := B_\eta \circ \Omega^{d-1}(A) \circ B_\eta^{-1}$

f) [Extra credit]Show that for any linear operator $A : V \to V$ we have $A \circ \tilde{A} = Det(A)Id$.

3) Let $A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$, $F(a, b, c, d, e, f) := Det(A)$. Show that F can be written in the form $F = P^2$ where P is a polynomial and find P.

4)Let A be an $n \times n$ matrix. For any $k, 1 \leq k \leq n$ we denote by A_k the $k \times k$ matrix which consists of the first k rows and columns of A. Show that if $Det(A_k) \neq 0$ for any $k, 1 \leq k \leq n$ then we can write A in the form $A = BC$ where B is lower triangular matrix and C is an upper unipotent matrix [that is A is an upper triangular matrix such that all the diagonal elements are equal to 1.

A hint. Use the row reduction procedure in the form it was discussed in the class.