Orientation on a vector space.

Let V be a vector space of dimension r. Then $\Omega^r(V)$ is a 1-dimesional space. Therefore $\Omega^r(V) - 0$ is a union of two rays. An orientation on V is a choice of one of these rays.

Let $B = \{e_1, ..., e_r\} \subset V$ be a basis and $\{e^1, ..., e^r\} \subset V^\vee$ the dual basis. Then $\omega_B := e^1 \wedge ... \wedge e_r$ is a non zero element of $\Omega^r(V)$. We denote by $\mathcal{O}_B$ the orientation on V given by the ray in $\Omega^r(V) - 0$ containing ω_B.

Let $B' = \{e'_1, ..., e'_r\}$ be another basis of V and $A : V \to V$ the linear map such that $e'_i = A(e_i), 1 \leq i \leq r$. Then (?) $\omega_B = Det(A)\omega_{B'}$. In particular we see that the orientation $\mathcal{O}_B$ corresponding to the basis B coinsides with the orientation $\mathcal{O}_{B'}$ corresponding to the basis B' if and only if $Det(A) > 0$. Therefore this definition of an orientation on V coincides with the one given in the class.

Let V, W be vector spaces of the same dimenstion $r, \mathcal{O}_V, \mathcal{O}_W$ orientations on V, W and $A : V \to W$ a linear map which defines an isomorphism between V and W. We say that A *is compatible with orientations* $\mathcal{O}_V, \mathcal{O}_W$ on V, W if $\Omega^r(A) : \Omega^r(W) \to \Omega^r(V)$ maps the ray $\mathcal{O}_W$ onto the ray $\mathcal{O}_V$.

Remark 1. If B_V, B_W are bases in V and W such that $\mathcal{O}_V = \mathcal{O}_{B_V}$ and $\mathcal{O}_W = \mathcal{O}_{B_W}$ then A *is compatible with orientations* $\mathcal{O}_V, \mathcal{O}_W$ on V, W if and ony if $Det(M_A) > 0$ where M_A is the matrix of the map A coresponding to the bases B_V and B_W.

In the case when $V = \mathbb{R}^r$ we denote by $\mathcal{O}_{st}$ the orientation on V corresponding to the standard basis $e_1, ..., e_r$ of $\mathbb{R}^r$.

Differential forms on a vector space.

Let V be a vector space of dimension m and $U \subset V$ an open subset. By the definition a differential r-form ω on U is a smooth function on U with values in the space $\Omega^r(V)$. We denote the space of r-forms on U by $\tilde{\Omega}^r(U)$.

Let $\mathcal{O}$ be an orientation of V and $\omega \in \tilde{\Omega}^m(U)$ an m-form with compact support. We define $\int_U \omega$ in the following way.

a) Choose any basis $B = \{e_1, ..., e_m\} \subset V$ of V such that $\mathcal{O}_B = \mathcal{O}$.

b) Define the smooth function f on U by $f(x^1, ...x^m) := \omega(u)(e_1, ..., e_m)$ for $(x^1, ...x^m) \in U$.

c) Define the integral $\int_U \omega$ by $\int_U \omega := \int_U f(x^1, ...x^m)dx^1...dx^m$.

Lemma 1. The integral $\int_U \omega$ is well defined [that is the integral $\int_U \omega$ does not depend on a choice of a basis B].

Let V be a vector space of dimension n and $e_1, ..., e_n$ a basis of V and $e^1, ..., e^n$ be the dual basis. We denote by $x_i, 1 \leq i \leq n$ linear functions

on V given by $x_i(v) := e^i(v)$. Since x_i is a function on $V, dx^i \in \tilde{\Omega}^1(V)$ is a 1-form on V. In other words dx_i is a function on V with values in $V^\vee$ Please check that for any $u \in V, dx_i(u) = e^i$. In other words dx_i is a "constant" 1-form. But be aware that the notion of a "constant" 1-form is not intrinsic-it depends of a choice of structure of a linear space on V. In other words if $\phi : V \to V$ is a smooth map and $\omega in \tilde{\Omega}^1(V)$ is a "constant" form the form $\phi^*(\omega)$ is not necesseraly "constant".

As in the problem set 2 we denote by $\mathcal{A}(r, n)$ be the set of subsets I of $[1, ..., n]$ such that $|I| = r$. For any $I = \{i_1 < ... < i_r\} \in \mathcal{A}(r, n)$ we associate an r-linear form ω^I on V by

$$\omega^I(v_1, ..., v_r) := \sum_{\sigma \in S_r} sign(\sigma) \prod_{1 \le k \le r} e^{i_{\sigma(k)}}(v_k)$$

.

It is easy to check that for any $I = \{i_1 < ... < i_r\} \in \mathcal{A}(r, n), v \in V$ we have $dx^{i_1} \wedge ... \wedge dx^{i_r}(v) = \omega^I$

As you know [see the problem set 2] the set $\omega^I, I \in \mathcal{A}(r, n)$ is a basis of the space $\Omega^r(V)$. Therefore for any open set $U \subset \mathbb{R}^n$ and any differensial form $\omega \in \tilde{\Omega}^r(U)$ we can write ω as a sum

$$\omega = \sum_{I = \{i_1 < ... < i_r\} \in \mathcal{A}(r, n)} f_I dx^{i_1} \wedge ... \wedge dx^{i_r}$$

where $f_I, I \in \mathcal{A}(r, n)$ are smooth functions on U. In particular any n-form ω on U has a form $\omega = f dx^1 \wedge ... \wedge dx^n$ where $f(u) = \omega(u)(e_1, ..., e_n)$.

Orientaion on manifolds.

Let $\Sigma \in \mathbb{R}^n$ be a smooth m-dimensional submanifold. By the definition an orientation $\mathcal{O}$ on Σ is a family of orientations $\mathcal{O}_\sigma$ on vector spaces $T_\Sigma(\sigma), \sigma \in \Sigma$ which *do not jump*. More precisely , we say that a family of orientations $\mathcal{O}_\sigma$ on vector spaces $T_\Sigma(\sigma), \sigma \in \Sigma$ *does not jump* if for any coordinate chart $\phi : U \to \Sigma, U \subset V$ the r-form $\phi^*(\omega)$ on U there exists an orientation $\mathcal{O}_V$ on V such that for any $u \in$ the linear map $D\phi(u) : V \to T_\Sigma(\phi(u))$ is compatible with the orientations $\mathcal{O}_\sigma$ on $T_\Sigma(\phi(u))$ and $\mathcal{O}_V$ on V.

Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two m-dimensional manifolds with orientations and $f : X \to Y$ a smooth map such that for any $x \in X$ the linear map $Df(x) : T_X(x) \to T_Y(y)$ is an isomorphism. We say that f *is compatible with the orientaions* on X and Y if for any $x \in X$ the linear map $Df(x) : T_X(x) \to T_Y(y)$ is compatible with the orientaions on the linear spaces $T_X(x)$ and $T_Y(y)$

Differential forms on a submanifold $\Sigma \subset \mathbb{R}^n$.

Let $\Sigma \in \mathbb{R}^n$ be a smooth m-dimensional submanifold. By the definition a differential r-form ω on Σ is "function" which associates to any point $\sigma \in \Sigma$ an r-form $\omega(\sigma) \in \Omega^r(T_\Sigma(\sigma))$ which varies "smoothly" with σ. More precisely let Ω be a "function" $\omega : (\sigma) \to \Omega^r(T_\Sigma(\sigma))$. We say that ω is *smooth* if for any coordinate chart $\phi : U \to \Sigma, U \subset V$ the r-form $\phi^*(\omega)$ on U is smooth where

$$\phi^*(\omega)(u)(v_1, ..., v_r) := \omega(\phi(u))(D\phi(u)(v_1), ..., D\phi(u)(v_r))$$

. We denote by $\tilde{\Omega}^r(\Sigma)$ the space of differential r-forms on Σ.

Manifolds with boundary.

Let Σ be a closed subset of $\mathbb{R}^n$. We say that Σ is a m-dimensional *submanifold with boundary* if there exists a closed subset $\partial\Sigma \subset \Sigma$ [called the bounday of Σ] such that

a) For any point $\sigma \in \Sigma - \partial\Sigma$ there exist an open neighbourhood U_σ of σ in $\mathbb{R}^n$, an open neighbourhood U_σ^0 of 0 in $\mathbb{R}^m$ and a smooth map $\phi_\sigma : U_\sigma^0 \to U \cap \Sigma$ which is onto, one-to-one , $\phi_\sigma(0) = \sigma$ and $D\phi(0) : \mathbb{R}^d \to \mathbb{R}^n$ is an imbedding. [In other words any point σ of $\Sigma - \partial\Sigma$ is smooth].

b) For any point $\sigma \in \partial\Sigma$ there exist an open neighbourhood U_σ of σ in Σ, an open neighbourhood U_σ^0 of 0 in $\mathbb{R}^m$ and a smooth map $\phi_\sigma : U_\sigma^0 \to \mathbb{R}^n$ such that $\phi_\sigma(0) = \sigma, D\phi(0) : \mathbb{R}^m \to \mathbb{R}^n$ is an imbedding and the restriction of ϕ_σ on $U_\sigma^0(+)$ is a map from $U_\sigma^0(+)$ to $U \cap \Sigma$ which is one-to-on and onto where $U_\sigma^0(+) = \{(x_1, ... x_m) \in U_\sigma^0 | x_m \geq 0\}$.

We call such a map ϕ_σ *a coordinate chart* for Σ at σ. I will leave for you to give a definition of the space $\tilde{\Omega}^r(\Sigma)$ for a manifold Σ with boundary. If $\phi_\sigma : U_\sigma^0 \to \Sigma \subset \mathbb{R}^n$ is a coordinate chart at σ we define the tangent space $T_\Sigma(\sigma) \subset \mathbb{R}^n$ as the image of the linear map $D\phi_i(0) : \mathbb{R}^m \to \mathbb{R}^n$.

I will leave for you to give a definition of the space $\tilde{\Omega}^r(\Sigma)$ of differential forms on manifolds with boundary.

Remark. By the definition [which you will give] for any $\omega \in \tilde{\Omega}^r(\Sigma), \sigma \in \partial\Sigma$ there exists $\omega^0 \in \tilde{\Omega}^r(U_\sigma^0)$ such that $\phi_\sigma^*(\omega)$ is equal to the restriction of ω^0 on $U_\sigma^0(+)$.

A partition of unity.

Let $\Sigma \subset \mathbb{R}^n$ be an m-dimensional submanifold with boundary and $\phi_i : U_i^0 \to U_i \subset \Sigma, 1 \leq i \leq N$ a family of coordinate charts such that $\cup U_i = \Sigma$. Then there exists a family of smooth functions f_i on Σ such that

a)$\sum_{1 \leq i \leq N} f_i = 1$
b)$supp f_i \subset U_i$.

Remark. We have shown the existence of a partition of unity for compact manifolds without a boundary but the same proof works for manifolds with boundary.

Integration of differential forms on manifolds with boundary.

Let Σ be a compact m-dimenstional manifold with boundary, $\omega \in \tilde{\Omega}^m(\Sigma)$ and $\mathcal{O}$ an orientation on Σ. To define the integral $\int_\Sigma \omega$ we have to

A) choose a finite set of coordinate charts $\phi_i : U_i^0 \to U_i, 1 \leq i \leq N$ such that

a) $\cup U_i = \Sigma$ and

b) The maps ϕ_i are compatible with the orientations on U_i^0 and U_i [when the orientation on U_i^0 is induced by the standard orientation on $\mathbb{R}^m$ and the orientation on U_i is induced by the orientation $\mathcal{O}$ on Σ].

B) Choose a partition of unity $f_i, 1 \leq i \leq N$ corresponding to the system ϕ_i of coordinate charts and

C) Define

$$\int_\Sigma^{\mathcal{O}, \phi_i, f_i} \omega := \sum_{1 \leq i \leq N} \int_{U_i^0} \omega_i$$

where $\omega_i := \phi_i^*(f_i \omega)$.

The main problem is to show that the $\int_\Sigma^{\mathcal{O}, \phi_i, f_i} \omega$ is well defined [that is does not depend on a choice of a system of coordinate charts ϕ_i and a partition of unity f_i].

A proof of the independence on a choice of a partition of unity follows almost immediately from the following result.

Claim 1. Assume that $supp\,\omega \subset U_i$. Then $\int_\Sigma^{\mathcal{O}} \omega(\phi_i, f_i) = \int_{U_i^0} \omega$.

I'll leave for you the proof of Claim 1.

Since we know already that $\int_\Sigma^{\mathcal{O}, \phi_i, f_i} \omega$ does not depend on a choice of a partition of unity f_i we will denote it by $\int_\Sigma^{\mathcal{O}, \phi_i} \omega$.

Proof of the independence on a choice of a system of coordinate charts.

Let $\psi_j : V_j^0 \to V_j \subset \Sigma, 1 \leq j \leq M$ be another system of coordinate charts for Σ.

Then we can consider a new system of coordinate charts $\theta_{i,j} : W_{i,j}^0 \to W_{i,j} \subset \Sigma, 1 \leq i \leq N, 1 \leq j \leq M$ where $W_{i,j} := U_i \cap V_j, W_{i,j}^0 := \phi^{-1}(W_{i,j}) \subset U_i^0 \subset \mathbb{R}^m$.

A proof of the independence on a choice of a system of coordinate charts follows almost immediately from the following result.

Claim 2. $\int_\Sigma^{\mathcal{O}, \phi_i} \omega = \int_\Sigma^{\mathcal{O}, \theta_{i,j}} \omega$.

I'll leave for you the proof of Claim 2.

Remark. I'll give now an "intuitive" way to define $\int_\Sigma \omega$. By the definition an m-form ω on an m-dimensional manifold Σ defines a way to measure a volume on a tangent space $T_\Sigma(\sigma)$ for any $\sigma \in bS$. If you cover Σ by small pieces $\Sigma = \cap U_j, 1 \le j \le N$ and choose points $\sigma_j \in U_j$ you can identify each piece U_j with a piece U_j^0 of a tangent space $T_\Sigma(\sigma_j)$ and define $\int_\Sigma \omega$ as a limit of partial sums $\sum_j volume(U_j^0)$ where you use $\omega(\sigma_j)$ to measure volume on $T_\Sigma(\sigma_j)$.

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be submanifolds and $\phi : X \to Y$ be a smooth map. Then for any $x \in X$ we have a linear map $D(\phi)(x) : T_X(x) \to T_Y(y), y = \phi(x)$. Therefore for any $r \ge 0$ we have a map $\phi_x^*[:= \Omega^r(D(\phi)(x))] : \Omega^r(T_Y(y)) \to \Omega^r(T_X(x))$. Using the maps $\phi_x^*, x \in X$ we define the map $\phi^* : \tilde{\Omega}^r(Y) \to \tilde{\Omega}^r(X)$ by $\phi^*(\omega)(x) := \phi_x^*(\omega_{(y)}), y = \phi(x)$. As we have seen for any $\omega' \tilde{\Omega}^{r'}, \omega'' \in \tilde{\Omega}^{r''}$ we have $\phi^*(\omega' \wedge \omega'') = \phi^*(\omega') \wedge \phi^*(\omega'')$.

If F is a smooth function on Y the chain rule $D(F \circ \phi)(x) = D(F)(y) \circ D(\phi)(x)$ shows that $\phi^*(dF) = d(\phi^*(F))$. Last time we proved the following result.

Proposition. Suppose A_r' and $A_r'', 0 \le r$ are two families of linear maps $A_r', A_r'' : \tilde{\Omega}^r(Y) \to \tilde{\Omega}^{r+1}(X)$ such that

a) $A_0'(F) = A_0''(F) = \phi^*(dF)$,

b') For any $\omega' \tilde{\Omega}^{r'}, \omega'' \in \tilde{\Omega}^{r''}$ we have $A_r'(\omega' \wedge \omega'') = A_{r'}'(\omega') \wedge \phi^*(\omega'') + (-1)^{r'} \phi^*(\omega') \wedge A_{r''}'(\omega'')$ and

b") For any $\omega' \tilde{\Omega}^{r'}, \omega'' \in \tilde{\Omega}^{r''}$ we have $A_r''(\omega' \wedge \omega'') = A_{r'}''(\omega') \wedge \phi^*(\omega'') + (-1)^{r'} \phi^*(\omega') \wedge A_{r''}''(\omega'')$.

Then $A_r' = A_r''$.

In the case when Y is an open domain in $\mathbb{R}^n$ we constructed explicitly a family of maps $A_r : \tilde{\Omega}^r(Y) \to \tilde{\Omega}^{r+1}(X)$ satisfying the conditions a) and b). we have defined the maps A_r by the formula $A_r(\omega) := \phi^*(d\omega)$.

Please show how the Proposition leads to a proof of the following result.

Lemma 2. For any smooth map $\phi : X \to Y$ the exists a family of operators $A_r : \tilde{\Omega}^r(Y) \to \tilde{\Omega}^{r+1}(X)$ satisfying the conditions a) and b).

Remark. If you consider the case when $Y = X$ and $\phi(x) = x$ you will obtain the definition of the differential $d_r : \tilde{\Omega}^r(X) \to \tilde{\Omega}^{r+1}(X)$.

Stokes's theorem. Let Σ be a compact m-dimensional manifold with the boundary $\partial\Sigma$, $\mathcal{O}$ an orientation on Σ and $\omega \in \tilde{\Omega}^{m-1}(\Sigma)$. Then

$$\int_\Sigma^{\mathcal{O}} d\omega = \int_{\partial\Sigma}^{\partial\mathcal{O}} \omega$$

.

To prove the Stokes's theorem we start with the following local result.

Lemma 3. Let $U^0 \subset \mathbb{R}^m$ be an open set, $\mathbb{R}^m_+ = \{x_1, ..., x_m\} | x_m \geq 0, \mathbb{R}^{m-1} = \{x_1, ..., x_m\} | x_m = 0$. Then for any differential form $\omega \in \tilde{\Omega}^{m-1}(U^0)$ on U^0 with compact support we have $\int_{U^0_+}^{\mathcal{O}_s t} d\omega = \int_{U^0 \cap \mathbb{R}^{m-1}}^{\mathcal{O}_s t} \omega$

.

Proof is completely analogous to the proof of the problem 3 in the problem set 6. Also was done in class in details.

To prove the Stokes's theorem for an arbitrary submanifold Σ of $\mathbb{R}^n$ we choose a system of coordinate charts $\phi_i : U^0_i \to U_i, 1 \leq i \leq N$ and a corrseponding partition of unity as in the section on "Integration of differential forms on manifolds with boundary". Since $\omega = \sum_{1 \leq i \leq N} \omega_i$ where $\omega_i := f_i \omega$ it is sufficient ot show that

$$\int_\Sigma^{\mathcal{O}} d\omega_i = \int_{\partial\Sigma}^{\partial\mathcal{O}} \omega_i$$

for all $i, 1 \leq i \leq N$. But by the definition

$$\int_\Sigma^{\mathcal{O}} d\omega_i = \int_{U^0_i}^{\mathcal{O}st} d\omega_i^0$$

and

$$\int_{\partial\Sigma}^{\partial\mathcal{O}} \omega = \int_{U^0_i \cap \mathbb{R}^{m-1}}^{\mathcal{O}_s t} \omega_i^0$$

.

Therefore the Stokes's theorem follows from the Lemma 3.