

Let  $A = (a_{i,j})$  be a symmetric  $n \times n$  matrix. We say that  $A$  is *positive* if for any vector  $\bar{x} \in \mathbb{R}^n, \bar{x} \neq \bar{0}$  we have  $\langle \bar{x}, A\bar{x} \rangle > 0$ . [In other words for any sequence  $(x_1, \dots, x_n) \neq (0, \dots, 0)$  we have  $\sum_{1 \leq i,j \leq n} a_{i,j}x_i x_j > 0$ ].

We say that  $A$  is *non negative* if for any vector  $\bar{x} \in \mathbb{R}^n$  we have  $\langle \bar{x}, A\bar{x} \rangle \geq 0$ .

1. Show that a  $2 \times 2$  matrix  $A$  is positive iff [if and only if]  $a_{11} > 0$  and  $Det(A) > 0$  where  $Det(A) := a_{11}a_{22} - a_{12}a_{21}$ .

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth [= infinitely differentiable] function. We say that a point  $u \in \mathbb{R}^n$  is *critical* if  $Df(u) = 0$ . We say that  $u$  is a point of *local minimum* if there exists an open set  $U \subset \mathbb{R}^n, u \in U$  such that for all  $u' \in U$  we have  $f(u') \geq f(u)$

We denote by  $Hess_f(u)$  a symmetric  $n \times n$  matrix  $H = (h_{i,j})$  where

$$h_{i,j} := \partial^2 f / \partial x_i \partial x_j (u)$$

. We call this matrix *the Hessian* of  $f$  at  $u$ .

2. a) Show that if  $u$  is a point of local minimum then the point  $u$  is critical and the Hessian  $Hess_f(u)$  is non negative.

b) Show that in the case when  $u$  is a critical point and the Hessian  $Hess_f(u)$  is positive  $u$  is a point of local minimum.

c) Given two positive numbers  $a, b, 0 < a < b$  find  $x_1, \dots, x_n, a < x_1 < \dots < x_n < b$  such that the fraction  $\frac{x_1 x_2 \dots x_n}{(a+x_1)(x_1+x_2)(x_2+x_3)\dots(x_n+b)}$  reaches it's maximum.

d) Show that the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) := (1 + e^y) \cos x - ye^y$  has infinite number of points of local maximum but does not have any points of local minimum.

3. Let  $\Sigma$  be a subset of  $\mathbb{R}^n, \sigma_0 \in \Sigma$ . In the previous homework I gave 3 definitions ( $Q1, Q2$  and  $Q3$ ) when  $\Sigma$  is a *smooth hypersurface* at  $\sigma_0$ .

a) Please give analogous definitions ( $Q1, Q2$  and  $Q3$ ) for a subset  $C \subset \mathbb{R}^3$  to be a *smooth curve* in  $\mathbb{R}^3$  at  $c \in \Sigma$  and show that these three definitions give the same result.

b) Let  $C \subset \mathbb{R}^3$  be a *smooth curve* in  $\mathbb{R}^3$  at  $c \in \Sigma$ . Give three definitions of the *tangent line*  $l_c$  and *tangent subspace*  $T_\Sigma(c) \subset \mathbb{R}^3$  corresponding to the three definitions ( $Q1, Q2$  and  $Q3$ ) of the *smoothness*. That is assume that  $(C, c)$  satisfies  $Qi, 1 \leq i \leq 3$  and give a definition of  $l_c$  and  $T_\Sigma(c)$  in terms of the definition  $Qi$ .

Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a smooth function such. We say that  $c \in \Sigma$  is a point of *local minimum* of the restriction of  $f$  on  $C$  if there exists an open set  $U \subset \mathbb{R}^n, u \in U$  such that for all  $c' \in U \cap C$  we have  $f(c') \geq f(c)$

c) Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a smooth function such that  $c \in \Sigma$  is a point of local minimum of the restriction of  $f$  on  $C$ . Show that the restriction of the differential  $D_f(c)$  to the tangent subspace  $T_\Sigma(c)$  is equal to zero.

d) Let  $C := S^2 \cap P$  where  $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ ,  $P := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1/2\}$ . Show that  $C$  is a smooth curve and find the point  $c \in C$  such that  $f(c) < f(c')$  for all  $c' \in C$  where  $f(x, y, z) := z$ .

4. Let  $A$  be a symmetric  $n \times n$  matrix,  $S^{n-1} := \{\bar{x} \in \mathbb{R}^n \mid \langle \bar{x}, \bar{x} \rangle = 1\}$ .

a) Show that for any  $s \in S^{n-1}$  we have  $T_{S^{n-1}}(s) = \{v \in \mathbb{R}^n \mid \langle v, s \rangle = 0\}$

Let  $f_A : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function such that  $f_A(v) := \langle Av, v \rangle$ .

b) Show that the restriction of the function  $f_A$  on  $S^{n-1}$  reaches its minimum at some point  $s \in S^{n-1}$ .

c) Let  $s \in S^{n-1} \subset \mathbb{R}^n$  be a point of maximum for the restriction of the function  $f_A$  on  $S^{n-1}$ . Show there exists a real number  $\lambda$  such that  $As = \lambda s$

d) Prove the existence of a basis  $v_1, \dots, v_n \in \mathbb{R}^n$  such that  $Av_i = \lambda_i v_i$  where  $\lambda_i, 1 \leq i \leq n$  are real numbers.