1) Explicitly, we need to determine when the expression $a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2$ is positive for all $(x_1, x_2) \neq (0,0)$. Clearly, we must have $a_{11} > 0$ for this to occur, and if $a_{11} > 0$ then this does occur whenever $x_2 = 0$. When $x_2 \neq 0$, we can rewrite this as requiring $a_{11}(x_1/x_2)^2 + (a_{12}+a_{21})(x_1/x_2) + a_{22} > 0$. As the leading coefficent of this parabola is positive, this will happen iff it has no real roots. This will happen iff $(a_{12}+a_{21})^2 - 4a_{11}a_{22} < 0$. As we are assuming A is symmetric, $a_{12} = a_{21}$ and $(a_{12}+a_{21})^2 = (2a_{12})^2 = 4a_{12}^2 = 4a_{12}a_{21}$ and our crtierion becomes that $a_{11}a22 - a12a_{21} = \det A > 0$.

2) In several parts of this problem, we will need to approximate functions by polynomials that match their derivatives. To that end, we prove the following result once and for all about this sort of situation: Let $U \subseteq \mathbf{R}^k$ be a convex set. Let $F : U \to \mathbf{R}$ be n-times differentiable, with the n^{th} derivative continuous. Let $u = (u_1, \ldots, u_k) \in \mathbf{R}^k$. For any $v = (v_1, \ldots, v_k) \in \mathbf{R}^k$, define the polynomial $P(v)$ by

$$P(v) = \sum_{i_1+\cdots+i_k \leq n} \frac{1}{i_1! \cdots i_k!} \frac{\partial^{i_1+\cdots+i_n}}{\partial x_1{}^{i_1} \cdots \partial x_k{}^{i_k}} F|_u (v_1 - u_1)^{i_1} \cdots (v_k - u_k)^{i_k}.$$

Let M be such that, for each $(i_1, \ldots, i_k)$ such that $i_1 + \cdots + i_k = n$,

$$|\frac{\partial^n}{\partial x_1{}^{i_1} \cdots \partial x_k{}^{i_k}} F(v) - \frac{partial^n}{\partial x_1{}^{i_1} \cdots \partial x_k{}^{i_k}} F(u)| < M.$$

(Those who like to keep track of lots of details can use a different bound of each $(i_1, \ldots, i_k)$.) Then

$$|F(v) - P(v)| \leq M \sum_{i_1+\cdots+i_k = n} \frac{1}{i_1! \cdots i_k!} |v_1 - u_1|^{i_1} \cdots |v_k - u_k|^{i_k}.$$

Proof: For simplicity, we assume $u = 0$. Let $G(v) = F(v) - P(v)$. Then we have

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1{}^{i_1} \cdots \partial x_k{}^{i_k}} G(0) = 0$$

and the result we want to show is that

$$|G(v)| \leq M \sum_{i_1+\cdots+i_k = n} \frac{1}{i_1! \cdots i_k!} |v_1|^{i_1} \cdots |v_k||^{i_k}.$$

Define a new function by $g(t) = G(tv_1, \ldots, tv_k)$. ($G$ is defined as U is convex.) It is clear that $(d^i/dt^i)g(0) = 0$ for $0 \leq i \leq n$. With a little more work, we get for any t,

$$(d^n/dt^n)g(t) = \sum_{i_1+\cdots+i_k = n} v_1{}^{i_1} \cdots v_k{}^{i_k} \frac{n!}{i_1! \cdots i_k!} \frac{\partial^{i_1+\cdots+i_n}}{\partial x_1{}^{i_1} \cdots \partial x_k{}^{i_k}} G(tv_1, \ldots, tv_n).$$

So

$$|(d^n/dt^n)g(t)| \leq M \sum_{i_1+\cdots+i_k = n} |v_1|^{i_1} \cdots |v_k|^{i_k} \frac{n!}{i_1! \cdots i_k!}.$$

Call the right hand of this expression K. So we know $|(d^n/dt^n)g(t)| \leq K$ and we want to prove $g(1) \leq K/n!$.

This can be done by repeated integration. We have

$$|(d^n/dt^n)g(t)| \leq K$$

so

$$|(d^{n-1}/dt^{n-1})g(t)| = |\int_0^t (d^n/dt^n)g(u)du| \leq \int_0^t K du = Kt$$

where we have used $(d^{n-1}/dt^{n-1})g(0) = 0$ to write the first equality. Repeating, we get $|(d^{n-k}/dt^{n-k})g(t)| \leq Kt^K/k!$, and conclude the result.

(For those of you who object that I shouldn't use integrals when they haven't been defined, I very quickly sketch a proof without them. Write $g(1) = L$. Define $h(t) = g(t) - Lt^n$. We have $h(0) = h(1) = 0$, so there is $c_1 \in [0, 1]$ with $(d/dt)h(c_1) = 0$. Also $(d/dt)h(0) = 0$. Repeating, we find $c_2 \in [0, c_1]$ with $(d^2/dt^2)h(c_2) = 0$. Continuing this way, we find there exists c_n such that $(d^n/dt^n)h(c_n) = 0$ or $(d^n/dt^n)g(c_n) = Ln!$. So $|L| = |d^n/dt^n)g(c_n)| \leq M/n!$, as desired.)

We now begin the problem. To simplify algebra, we take $u = 0$ in parts *(a)*, *(b)* and assume $f(u) = 0$.

a) We first show $D_f|_0$ is 0. If not, let v be such that $D_f|_0(v) = -a < 0$. As D_f is continuous, we can chose a convex neighborhood U of 0 so that $|D_f|_u(v) - D_f|_0(v)| \leq a/10$ for $u \in U$. Then by our lemma applied to U with $n = 1$, $|f(tv) - D_f|_0(tv)| = |f(tv) - t(-a)| \leq (a/10)t|$ for t small enough that $tv \in U$. But then $f(tv) < -9at/10$. As we have shown f is zero arbitrarily close to 0, this contradicts the claim that 0 is a local minimum.

We now prove $Hess_f(0)$ is nonnegative. If not, let v be such that $v^{\mathrm{T}} Hess_f(0)v = -a < 0$. Find U a convex neighborhood of 0 such that $|v^{\mathrm{T}} Hess_f(u)v - v^{\mathrm{T}} Hess_f(0)v| \leq a/10$. As before, we get using the lemma in the case $n = 2$ that $|f(tv) - (1/2)(tv)^{\mathrm{T}} Hess_f(0)(tv)| \leq (1/2)at^2/10$ so $f(tv) < -9(a/2)t^2/10$. (I leave it to you to check this really is the result of explanding the formula out.

b) Let S be the set of vectors $(v_1, \ldots, v_n)$ such that $\sum |v_i| = 1$. As S is bounded and closed it is compact. $v^{\mathrm{T}} Hess_f(0)v$ is a continuous function of v (it is given by a polynomial) so it attains its minimum value on S, let this value be a. By assumption, $0 < a$. Take a convex neighborhood U of 0 on which every derivative $\partial^2/\partial x_i \partial x_j$ obeys

$$|\partial^2/\partial x_i \partial x_j f|_u - \partial^2/\partial x_i \partial x_j f|_0| < \frac{a}{10}.$$

Then by our lemma again, for $v \in U$,

$$|f(v) - v^{\mathrm{T}} Hess_f(0)v^T| \leq \sum_{i,j} \frac{a}{10} v_i v_j = \frac{a}{10} (\sum_i v_i)^2.$$

Write $v = rw$ where $w \in S$, $r \geq 0$. If $v \neq 0$, $r > 0$ and we can write $f(v) \geq (rv)^{\mathrm{T}} Hess_f(0)(rv) - (a/10)r^2 \geq ar^2 - (a/10)r^2 = (9a/10)r^2 > 0. So f is positive on U.$

c) For simplicity of notation, set $x_0 = a$, $x_{n+1} = b$. This problem is awkward, because of the difficulty of making sure the maximum occurs at an interior point. Set

$$f(x_1, \ldots, x_n) = \frac{x_1 \cdots x_n}{(x_0 + x_1)(x_1 + x_2) \cdots (x_n + x_{n+1})}.$$

We first find the sole local maximum of f on the set $(\mathbf{R}^+)^n = \{(x_1, \ldots, x_n), x_i > 0\}$. Let $(x_1, \ldots x_n)$ be a local maximum, for every i from 1 to n we must have $(\partial f/\partial x_i) = 0$ or

$$(\frac{1}{x_i} - \frac{1}{x_i + x_{i-1}} - \frac{1}{x_i + x_{i+1}})f(x_1, \ldots, x_n) = 0.$$

As f is never 0 (we have all $x_i > 0$), we must have

$$\frac{1}{x_i} - \frac{1}{x_i + x_{i-1}} - \frac{1}{x_i + x_{i+1}} = 0$$

or, rearranging

$$(x_i + x_{i-1})(x_i + x_{i+1}) = x_i(2x_i + (x_{i-1} + x_{i+1}))$$

or

$$x_i^2 + x_i(x_{i-1} + x_{i+1}) + x_{i-1}x_{i+1} = 2x_i^2 + x_i(x_{i-1} + x_{i+1})$$

or

$$x_i^2 = x_{i-1}x_{i+1}$$

or, since $x_i \neq 0$,

$$x_{i+1}/x_i = x_i/x_{i-1}.$$

So x_i/x_{i-1} is a constant, call it c and we get $c^{n+1} = (x_{n+1}/x_n)(x_n/x_{n-1})\cdots(x_1/x_0) = x_{n+1}/x_0 = b/a$. So, if f has a local maximum on $(\mathbf{R}^+)^n$, this solution is given by $x_i = a^{i/(n+1)}b^{(n+1-i)/(n+1)}$. At this point,

$$f = (1/a)(a/(a+ac))(ac/(ac+ac^2))\cdots(ac^n/(ac^n+ac^{n+1})) = (1/a)/(1+c)^n.$$

We now begin the hard part of this problem, showing that this is in fact the global maximum. As in many problems like this, it is convenient to relax the restraints of the problem and perform our maximization on a larger domain. Let m_i be the point we are attempting to show the maximum is attained at, $m_i = a^{i/(n+1)}b^{(n+1-i)/(n+1)}$. We will show that this is actually the point on the entire set $\{x_i > 0\}$ where f takes its maximum. Let $f(m_1,\ldots,m_n) = \alpha > 0$. We will use m_0 and $m_{n|1}$ to refer to a and b respectively, just as we did x_0 and x_{n+1}.

Lemma: There are constants k and K so that if any $x_i > K$ or any $x_i < k$, we have $f(x_1,\ldots,x_n) < \alpha/2$.

Proof: Note that we have $x_i/(x_i + x_{i+1}) < 1$ and $x_i/(x_{i-1}+x_i) < 1$ for every i. Therefore we can write

$$f(x_1,\ldots,x_n) = \frac{x_1}{x_0+x_1}\frac{x_2}{x_1+x_2}\cdots\frac{x_i}{x_{i-1}+x_i}\frac{1}{x_i+x_{i+1}}\frac{x_{i+1}}{x_{i+1}+x_{i+2}}\cdots\frac{x_n}{x_n+x_{n+1}}$$

so

$$f(x_1,\ldots,x_n) < 1\cdot 1\cdots 1\cdots\frac{1}{x_i+x_{i+1}}\cdots 1\cdots 1 < \frac{1}{x_i}.$$

Thus we can take $K = 2/\alpha$. Similarly

$$f(x_1,\ldots,x_n) = \frac{1}{a+x_1}\frac{x_1}{x_1+x_2}\cdots\frac{x_{i-1}}{x_{i-1}+x_i}\frac{x_i}{1}\frac{x_{i+1}}{x_i+x_{i+1}}\cdots\frac{1}{x_n+b}$$

so

$$f(x_1,\ldots,x_n) < \frac{1}{a}\cdot 1\cdots 1\cdot x_i\cdot 1\cdots\frac{1}{b} = \frac{x_i}{ab}.$$

Thus we can take $k = ab\alpha/2$.

Given this lemma, define $S \subset \mathbf{R}^n$ as $\{k \le x_i \le K\}$. S is compact, so f assumes its maximum at some point in it. For $(x_1,\ldots,x_n) \notin S$, we have $f(x_1,\ldots,x_n) < \alpha/2$, where as $f(m_1,\ldots,m_n) = \alpha$ so $(m_1,\ldots,m_n) \in S$ (this can also be shown directly) and $f(m_1,\ldots,m_n) > f(x_1,\ldots,x_n)$ for $(x_1,\ldots,x_n) \notin S$. Moreover, for any $(x_1,\ldots,x_n)$ on the boundary of S, we also have $f(x_1,\ldots,x_n) < \alpha/2 < f(m_1,\ldots,m_n)$ so f can not take it's maximum value on S anywhere on the boundary. So the point where it assumes it's maximum is an interior point and thus a local maximum of f on $\mathbf{R}^n$. We have shown $(m_1,\ldots,m_n)$ is the only such point.

d) The critical points of f are given by requiring first that $\partial f/\partial x = 0$ so $e^y(-\sin x) = 0$ and $x = n\pi$. We also require $\partial f/\partial y = e^y(\cos(n\pi) - 1 - y) = 0$ so $(-1)^n - 1 - y = 0$. We thus get that all dritical points are of the form $(n\pi, (-1)^n - 1)$.

The general Hessian is

$$\begin{pmatrix} -e^y\cos x & -e^y\sin x \\ -e^y\sin x & e^y(\cos x - y - 2)\end{pmatrix}.$$

We can factor out e^y with out effecting the positiveness of our matrix, as $e^y > 0$. PLugging in our formula for the general critical point to what's left, we get

$$\begin{pmatrix} -1 & 0 \\ 0 & 1-((-1)^n-1)-2\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & (-1)^n\end{pmatrix}.$$

This expression is negative for n odd, this giving a local maximum there, but never positive.

3) a) In each definition, $C \subseteq \mathbf{R}^3$, $c \in C$, we are defining what it means for C to be smooth at c. The coordinates of c are (c^1, c^2, c^3).

Q1) There is an i, $i \in \{1,2,3\}$ an open set $U \subset \mathbf{R}$ with $c^i \in U$ and smooth functions $f_j, f_k : U \to \mathbf{R}$ where j and k are the two elements of $\{1,2,3\}$ other than i such that, if $f : U \to \mathbf{R}^3$ is defined by $f(t)^i = t$, $f(t)^j = f_j(t)$ and $f(t)^k = f_k(t)$, we have an open subset $V \subset \mathbf{R}^3$ with $c \in V$ and $V \cap C = V \cap f(U)$.

Q2) There is an open set $U \subset \mathbf{R}^3$ with $c \in U$ and a smooth function $F : \mathbf{R}^3 \to \mathbf{R}^2$ such that $U \cap C = \{v \in \mathbf{R}^3$ such that $F(v) = G(v) = 0\}$ and $D_F(c)$ is onto.

Q3) There is an open set $U \in \mathbf{R}$, $0 \in U$ with a smooth map $\phi : U \to \mathbf{R}^3$ such that $\phi(0) = c$, $D_\phi(0) \neq 0$ and, for any open set $V \subset U$, $0 \in V$, there exists open $W \subset \mathbf{R}^3$, $c \in W$ such that $W \cap C = \phi(V) \cap W$.

(These can be rewritten to use balls instead of general open sets, but there is no need.)

We now show they are equivalent (this was not required)

$1 \Leftarrow 2$: for simplicity, assume $i = 1$, $j = 2$ and $k = 3$, the other cases are similar. Take $F(x^1, x^2, x^3) = (x^2 - f_2(x^1), x^3 - f_3(x^1))$. We can take the U of definition 2 to be the V of definition 3. The only hard part is checking that the map is onto. The matrix of $D_F(c)$ is

$$\begin{pmatrix} f_2'(x^1) & 1 & 0 \\ f_3'(x^1) & 0 & 1 \end{pmatrix}.$$

As the image contains $\binom{0}{1}$ and $\binom{1}{0}$, the map is onto.

$1 \Leftarrow 3$: Again, assume for simplicity $i = 1$, $j = 2$, $k = 3$ and also take $c = (0,0,0)$. Set $\phi(t) = (t, f_2(t), f_3(t))$ and take the U of definition 3 to be the same as in definition 1. We first show $D_\phi(0) \neq 0$. This is easy, as $\partial\phi/\partial t = 1$. We now show, for any open $V \subset U$, $c \in U$, we can find $W \subset \mathbf{R}^3$ such that $\phi(V) \cap W = C \cap W$. By the assumptions of defintion 1, we have $\phi(U) = C$ so $\phi(V) \subseteq C$ and $\phi(V) \cap W \subseteq C \cap W$. All we have to do is find a W such that, if $\phi(s) \in W$, $s \in V$, this will prove the reverse inclucion. Take W to be the set of points in U with x^1 coordinate in U. As the x^1 coordiinate of $\phi(s)$ is s, this suffices.

$2 \Leftarrow 1$: $D_F(c)$ is a 3X2 matrix that is required to have it's columns span $\mathbf{R}^2$. For this to happen, two of them must be linearly independent, without loss of generality, the last two. What we are trying to show is now precisely Theorem 9.5.1 in our text book, with $n = 2$, $k = 1$. (This result was proven in class, as well.)

$3 \Leftarrow 1$: Write ϕ as $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$. As $D_\phi(0) = (\phi_1'(0), \phi_2'(0), \phi_3'(0)) \neq 0$, some $\phi_i'(0) \neq 0$, say without loss of generality $\phi_1'(0) \neq 0$. Let $p : \mathbf{R}^3 \to \mathbf{R}$ and $q : \mathbf{R}^3 \to \mathbf{R}$ be the maps of projecting onto the first and onto the last two coordinates respectively, so $\phi_1 = p \circ \phi$. By the inverse function theorem, there exists an open set $U \subset \mathbf{R}$ containing $\phi_1(0)$ and a function $g : U \to \mathbf{R}$ such that g and $\phi_1 = p \circ \phi$ are inverses, that is, $g \circ p \circ \phi$ and $p \circ \phi \circ g$ are the identity.

We now claim the image of g is open. We need to show for any $x \in U$, there is an open interval about $g(x)$ lying in $g(U)$. By the inverse function theorem, $g'(x) = 1/phi_1'(g(x)) =\neq 0$, let $g'(x) = a$ and assume without loos of generality $a > 0$. By the continuity of g', we can find an interval $(x - \delta, x + \delta) \subset U$ on which $g' > a/2$. Then, by problem 1 of set 1, $(g(x) - (a/2)\delta, g(x) + (a/2)\delta) \subset g((x - \delta, x + \delta)) \subset g(U)$ so we can take as our interval $(x - (a/2)\delta, x + (a/2)\delta)$. Set $V = g(U)$.

Take $W \in \mathbf{R}^3$ open, containing c, such that $\phi(V) \cap W = C \cap W$. Define $f(t) = (t, q \circ \phi \circ g(t))$. We claim $W \cap C = W \cap f(U)$, this will prove the result. (There is a little stupid detail needed here about projecting f unto the second and third coordinate to produce f_2 and f_3.)

First, note we can rewrite $f(t)$ as $(p \circ \phi \circ g(t), q \circ \phi \circ g(t)) = \phi \circ g(t)$ (Remember p and q are just projection onto the first and the last two coordinates respectively.) So we are to show $W \cap C = W \cap \phi \circ g(U) = \phi(V)$. But we constructed W to have precisely this property.

b) Q1) If i is 1, the tangent space is given by the set of all points of the form $t(1, f_2'(c), f_3'(c))$. If i is not 1, make the obvious rearrangement.

Q2) The tangent space is given the kernel of D_F (as D_F has two dimensional image, it has one dimensional kernel.)

Q3) The tangent space is the set of all points of the form $t(\phi_1'(0), \phi_2'(0), \phi_3'(0)) = D_\phi(0)(t)$.

In each case, the tangent line is given by translating each of these spaces by c, e.g., in the first definition it would be the set of points of the form $c + t(1, f_2'(c), f_3'(c))$.

c) We use defintion Q3. Let U be an open set of $\mathbf{R}^3$ containing c. By the continuity of ϕ, there is n open set V in $\mathbf{R}$ containing 0 such that $\phi(V) \subset U$. For all $c' \in U \cap C$, $f(c') \geq f(c)$. For any $t \in V$, we have $\phi(t) \in U \cap C$ and we have $\phi(0) = c$, so for $t \in V$, $f(\phi(t)) \geq f(\phi(0))$ so $f \circ \phi$ has a local minimum at 0. Using the result from 1-variable calculus, $D_f(c)D_\phi(0) = 0$. The image of $D_\phi(0)$ is preciesly the tangent space at c, so we get $D_f(c)$ is 0 restricted to the tanget space at c.

d) We first show C can be defined in terms of defintion Q2 by the function $F(x, y, z) = (x^2 + y^2 + z^2 - 1, x + y + z - 1)$. The only worry is that D_F might not have two dimensional image. At x_0, y_0, z_0, D_F is the map $(a, b, c) \to (2x_0a + 2y_0b + 2z_0c, a + b + c)$. The only way this does not have two dimensional image is if the two coordinates are always proportional, in which case $2x_0/1 = 2y_0/1 = 2z_0/1$ so $x_0 = y_0 + z_0$, call this common value k. But then we must have both $3k = 1/2$ and $3k^2 = 1$, two equations with no common root.

We now solve the problem. We know some such c exists as C is compact and f continuous, we just need to find it. Let $f(x, y, z) = z$, by the previous problem, D_f is zero restricted to the tangent space at c, which is just the kernel of D_F. Write $c = (x_0, y_0, z_0)$, D_f is just the map $(a, b, c) \to c$. We want to discover when $\{(a, b, c) : c = 0\}$ contains $\{(a, b, c) : a + b + c = 2x_0a + 2y_0b + 2z_0c = 0\}$. As the latter space is 1-dimensional, (we just showed the image of D_F is 2-dimensional, so it's kernel is 1-dimensional) we have that the former space contains the latter iff they have nonzero intersection. Let $(a, b, 0)$ by this intersection. We need a nonzero solution to $a + b = 2x_0a + 2y_0b = 0$. The first equation implies $a = -b$ which then gives $a(2x_0 - 2y_0) = 0$, as we want a nontrivial solution, $a \neq 0$ and $x_0 = y_0$.

So what are the points on C with $x_0 = y_0$? They are of the form $(x, x, 1/2 - 2x)$. But we also need $x^2 + x^2 + (1/2 - 2x)^2 = 1$ or $6x^2 - 2x - 3/4 = 0$. This is a quadratic with roots $(2 \pm \sqrt{22})/12$. A little investigation show that it is the positive rot that gives a minimum and the positive root that gives a maximum. So the answer is

$$\left(\frac{2+\sqrt{22}}{12}, \frac{2+\sqrt{22}}{12}, \frac{1-\sqrt{22}}{6}\right)$$

.

4) a) Using definition Q2 of a hypersurface, S is given by $F(v) =< v, v > -1$. We have $D_F(s) : v \to< v, s > + < s, v >= 2 < v, s >$. (Write it out in cordinates if you don't believe it.) So the tangent space is the kernel of this map, $\{s :< v, s >= 0\}$.

b) f_A is continuous and S is compact, so this is automatic.

c) We have $D_f(s) : v \to< s, Av > + < v, As >$. But A is symmetric, so this is just $2 < v, As >$. We showed that, if s is a local maximum, D_f is 0 restricted to the tangent space of S at s, so we want $< v, s >= 0$ to imply $< v, As >= 0$. (Note, the reverse implication may not hold, as we may have $As = 0$, we never required A to be invertible.) As $s \in S$, $s \neq 0$. This can only occur if $As = \lambda s$ for some s.

d) Proof by induction on n (the dimension of the vector space). For $n = 1$, this is obvious. If A is a symmetric linear operater from $\mathbf{R}^n \to \mathbf{R}^n$, by the above, there is v_n and λ_n such that $Av_n = \lambda v_n$. Let $v_n^\perp$ be $\{w :< w, v_n >= 0\}$. For any w, we have $< Aw, v_n >=< w, Av_n >=< w, \lambda v_n >= \lambda < w, v_n >$ so A maps $v_n^\perp$ to itself. Let $e_1, \dots, e_{n-1}$ be an orthonormal basis of $v_n^\perp$, we claim A restricted to $v_n^\perp$ is symmetric when written in the basis e_i.

Let $\tilde{A}$ be this matrix, we have $\tilde{A}_{ij} =< e_i, \tilde{A}e_j >=< e_i, Ae_j >$. As A is symmetric, $< Ae_i, e_j >=< e_i, Ae_j >$ so the conclusion is proved.

We can now apply the induction hypothesis to $\tilde{A}$ and find a basis $v_1, \dots, v_{n-1}$ of $v_n^\perp$ and real numbers $\lambda_1, \dots, \lambda_{n-1}$ with $\tilde{A}v_i = \lambda v_i$. We have $Av_i = \tilde{A}v_i$, so all we need to show is that $v_1, \dots, v_n$ is a basis of $\mathbf{R}^n$. As there are n of them, it suffices to show linear independence. By induction, $v_1, \dots, v_{n-1}$ are indepedent, so we need to show there is no relation of the form $v_n = a_1v_1 + \cdots + a_{n-1}v_{n-1}$. The right hand side of this expression is clearly in $v_n^\perp$, so this claim just ammounts to $< v_n, v_n >\neq 0$. As $v_n \neq 0$, this is trivial.

(Note: we have not only shown the v_i exist, we have shown they are orthogonal.)