1. Find for which real numbers a the integral $\int_0^{+\infty} \frac{e^{ax}}{1+x^2} dx$ is convergent? [That is there exists a limit $\lim_{N\to\infty} \int_0^N \frac{e^{ax}}{1+x^2} dx$

Since the function $\frac{e^{ax}}{1+x^2} dx$ is positive the integral $\int_0^{+\infty} \frac{e^{ax}}{1+x^2} dx$ is convergent iff the sequence $c_N(a) := \int_0^N \frac{e^{ax}}{1+x^2} dx$ is bounded when $N \to \infty$.

Consider first the case when $a = 0$. Then we have

$$\int_0^N \frac{1}{1+x^2} dx = \int_0^1 \frac{1}{1+x^2} dx + \int_1^N \frac{1}{1+x^2} dx$$

. So the sequence c_N is bounded iff the sequnce $d_N := \int_1^N \frac{1}{1+x^2} dx$ is bounded. But $frac11 + x^2 < 1/x^2$. So $d_N < \int_1^N 1/x^2 dx = 1-1/N < 1$. So the integral $\int_0^{+\infty} \frac{e^{ax}}{1+x^2} dx$ is convergent for $a = 0$.

If $a < 0$ then $\frac{e^{ax}}{1+x^2} < \frac{1}{1+x^2}$. Therefore $c_N(a) < c_N(0)$ and we see that the sequence $c_N(a) := \int_0^N \frac{e^{ax}}{1+x^2} dx$ is bounded for $a \leq 0$.

If $a > 0$ then it is easy to check that $\frac{e^{ax}}{1+x^2} \to \infty$ for $x \to \infty$. therefore the integral $\int_0^{+\infty} \frac{e^{ax}}{1+x^2} dx$ is divergent for $a > 0$.

Remark. You could compute the indefinite integral $\int_0^N \frac{1}{1+x^2} dx$ explicitely but I want to show how to estimate integrals even when one can not compute them explicitely.

2. a)Use Fubini theorem to express the integral

$$\int_0^1 \pi(\int_y^\pi \frac{\sin x}{x} dx) dy$$

as a double integral.

b)Write this integral as an iterated integral in the other order.

c)Compute the integral

3.What is the area of the ellipse

$$x^2/a^2 + y^2/b^2 \leq 1$$

[a,b are given positive numbers]

A solution.. Use thee change of variables $u = x/a, v = y/b$.

Then the image of the ellipse $x^2/a^2 + y^2/b^2 \leq 1$ is the circle

$$u^2 + v^2 \leq 1$$

Since $Det(D\phi(x,y)) = ab$ for all $(x,y) \in \mathbb{R}^2$ we see that

area of the ellipse=ab (area of the circle $u^2 + v^2 \leq 1$)=$ab\pi$

4.Find the volume between the cone of equation $z^2 = x^2 + y^2$ and the paraboloid of equation $z = x^2 + y^2$

A hint. Use the coordinate z and the polar coordinates r, θ on the plane x, y. Then you will see that the integral is equal to

$$2\pi \int_0^1 r dr \left(\int_{r^2}^r dz \right) = 2\pi \int_0^1 (r^2 - r^3) dr = \pi/6$$