Let $A = (a_{i,j})$ be a symmetric $n \times n$ matrix. We say that A is *positive* if for any vector $\bar{x} \in \mathbb{R}^n, \bar{x} \neq \bar{0}$ we have $< \bar{x}, A\bar{x} >> 0$. [In other words for any sequence $(x_1, ..., x_n) \neq (0, ..., 0)$ we have $\sum_{1 \leq i,j \leq n} a_{i,j} x_i x_j > 0$].

We say that A is *non negative* if for any vector $\bar{x} \in \mathbb{R}^n$ we have $< \bar{x}, A\bar{x} > \geq 0$.

Remark. The language is tricky. If A is not *non negative* it does not imply that it is negative.!!!

1. Show that a 2×2 matrix A is positive iff [if and only if] $a_{11} > 0$ and $Det(A) > 0$ where $Det(A) := a_{11}a_{22} - a_{12}a_{21}$.

Since everyone solved the problem I'll not write up a solution.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth [= infinitely differentiable] function. We say that a point $u \in \mathbb{R}^n$ is *critical* if $D_f(u) = 0$. We say that u is a point of *local minimum* if there exists an open set $U \subset \mathbb{R}^n, u \in U$ such that for all $u' \in U$ we have $f(u') \leq f(u)$

We denote by $Hess_f(u)$ a symmetric $n \times n$ matrix $H = (h_{i,j})$ where

$$h_{i,j} := \partial^2 f / \partial x_i \partial x_j (u)$$

. We call this matrix *the Hessian* of f at u.

2. a)Show that if u is a point of local minimum then the point u is critical and the Hessian $Hess_f(u)$ is non negative.

Proof. To prove that the point u is critical we assume that it isn't an show that this assumption leads to a contrudiction. So assume that $D_f(u) \neq 0$. Then there exists $\bar{v} \in \mathbb{R}^n$ such that $D_f(u)(\bar{v}) \neq 0$. Consider the function $F : \mathbb{R} \to \mathbb{R}, F(t) := f(u + t\bar{v})$. Since $F'(t) = D_f(u + t\bar{v})(\bar{v})$ (?) it foollows from the Taylor's formula we see that

$$F(t) - F(0) - tD_f(u)(\bar{v}) = t^2/2F''(c), c \in [0, t]$$

Since "for small t, t^2 is *much smaller* then t" and $D_f(u)(\bar{v}) \neq 0$ we can find small t [negative if $D_f(u)(\bar{v}) > 0$ and positive if $D_f(u)(\bar{v}) < 0$] such that $F(t) - F(0) < 0$. That is $f(u + t\bar{v}) - f(u) < 0$. A contrudiction.

More formally. Since the function $F'(t)$ is continuous we can find $\epsilon > 0$ such that for all $t, |t| < \epsilon$ the function $F'(t) \neq 0$ and has the same sign as $D_f(u)(\bar{v})$. Assume that $D_f(u)(\bar{v})$ is positive. Choose any $t, -\epsilon < t < 0$. Then by the MVT we have $F(t) - F(0) = tF'(c)$ for some $c, t \leq c \leq 0$. So $F(t) - F(0) < 0$. That is $f(u + t\bar{v}) - f(u) < 0$. A contrudiction.

To finsih the proof of a) we have to show that the assumption that the Hessian $Hess_f(u)$ is not non negative.If $Hess_f(u)$ is not non negative the we can find $\bar{v} \in \mathbb{R}^n$ such that $< Hess_f(u)(\bar{v}), \bar{v} >< 0$. Define the function $F : \mathbb{R} \to \mathbb{R}, F(t) := f(u + t\bar{v})$. Then $F''(0) =$

$Hess_f(u)(\bar{v}), \bar{v} >$(?). Since the function $F''(t)$ is continuous we can find $\epsilon > 0$ such that for all $t, |t| < \epsilon$ the function $F''(t) > 0$. Fix $t, 0 < t < \epsilon$. By the Taylor's formula we have [since $F'(0) = 0$]

$$F(t) - F(0) = t^2/2F''(c)$$

for some $c, t \leq c \leq 0$. So $F(t) - F(0) < 0$. A contrudiction.

b)Show that in the case when u is a critical point and the Hessian $Hess_f(u)$ is positive u is a point of local minimum.

The same arguments as in a) give the proof.

c) Given two positive numbers $a, b, 0 < a < b$ find $x_1, ..., x_n, a < x_1 < ... < x_n < b$ such that the fraction $\frac{x_1 x_2 ... x_n}{(a+x_1)(x_1+x_2)(x_2+x_3)...(x_n+b)}$ reaches it's maximum.

Proof. Let $C \subset \mathbb{R}^n$ be the symplex $yC = \{(x_1, ..., x_n) | a \leq x_1 \leq ... \leq x_n \leq b\}$ and $F : C \to \mathbb{R}$ be the function $F(x_1, ..., x_n) := \frac{x_1 x_2 ... x_n}{(a+x_1)(x_1+x_2)(x_2+x_3)...(x_n+b)}$. It is easy to see that C is compact and F is continuous. Therefore F reaches it's minimum at some point $\bar{u} = (u_1, ..., u_n) \in C$. We define $u_0 := a, u_{n+1} := b$. One checks (?) that the following result is true.

Lemma.$\partial F/\partial x_i(u) = 0 \Leftrightarrow u_{i-1}u_{i+1} = u_i^2, 1 \leq i \leq n$. Moreover $\partial F/\partial x_i(u) > 0 \Leftrightarrow u_{i-1}u_{i+1} > u_i^2$ and $\partial F/\partial x_i(u) < 0 \Leftrightarrow u_{i-1}u_{i+1} < u_i^2, 1 \leq i \leq n$.

Let ∂C be the boundary of the symplex C.

Claim $\bar{u}$ does not belong to ∂C.

Proof. Assume that $\bar{u} \in \partial C$ and come to a contrudiction. If $\bar{u} \in \partial C$ then in one of the inequalities defining C we have an equality. That is either $u_1 = a$ or $u_n = b$ or for some $i, 1 \leq i \leq n-1$ we have $u_i = u_{i+1}$. In all these cases you can find $i, 0 \leq j \leq n$ such that $u_j \neq u_{j+1}$ but either $j > 0$ and $u_{j-1} = u_j$ or $j < n$ and $u_{j+1} = u_j$. In such a case it follows from Lemma that $\partial F/\partial x_j(u) > 0$. Since you move the point $\bar{u}$ by you increasing u_j we are staying inside C you see that $\bar{u}$ is not a point of minimum for F. The Claim is proved.

So we know that $\bar{u} \in C$. Since $\bar{u}$ is a critical point for F we see that $u_{i-1}u_{i+1} = u_i^2, 1 \leq i \leq n$. Since $u_i > a > 0$ we can define $z_i := \log u_i$. Then we have $2z_i = z_{i-1} + z_{i+1}, 1 \leq i \leq n, z_0 = \log a, z_n = \log b$. I'll leave for you to solve this sustem of linear equations for $z_i, 1 \leq i \leq n$.

d)Show that the function $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) := (1 + e^y)\cos x - ye^y$ has infinite number of points of local maximum but does not have any points of local minimum.

Straightforward.

3. Let Σ be a subset of $\mathbb{R}^n, \sigma_0 \in \Sigma$. In the previous homework I gave 3 definitions ($Q1, Q2$ and $Q3$) when Σ is a *smooth hypersurface* at σ_0 .

a) Please give analogous definitions ($Q1, Q2$ and $Q3$) for a subset $C \subset \mathbb{R}^n$ to be a *smooth curve* in $\mathbb{R}^n$ at $c \in \Sigma$ and show that these three definitions give the same result.

Solution.

Let Σ be a subset of $\mathbb{R}^n, \sigma_0 \in \Sigma$. We say that Σ is *smooth curve* at σ_0 if one of the following three conditions $Q1, Q2, Q3$ is satisfied.

Let $\sigma_0 = (x_0^i), 1 \leq i \leq n$. We define $\sigma_0^i \in \mathbb{R}^{n-1}$ by $\sigma_0^i := (x_0^1, ..., x_0^{i-1}, x_0^{i+1}, ..., x_0^n)$.

$Q1$.There exists $\epsilon > 0, i, 1 \leq i \leq n$ an open set $U \subset \mathbb{R}, x_0^i \in U$ and a smooth functions $f_j, 1 \leq j \leq n, j \neq i : U \to \mathbb{R}$ such that

A)$f_j(x_0^i) = x_0^j$

B) for any $x = x \in U$ we have $(f_1(x), ..., f_{i-1}(x), x, f_{i+1}(x), ..., f_n(x)) \in \Sigma$ and

C)there exists an open set $V \subset \mathbb{R}^n, \sigma_0 \in V$ such that for any $\sigma \in \Sigma \cap U$ there exists $x \in U$ such that $\sigma = (f_1(x), ..., f_{i-1}(x), x, f_{i+1}(x), ..., f_n(x)) \in \Sigma$

$Q2$. There exists an open set $U \subset \mathbb{R}^n, \sigma_0 \in U$ and a smooth function $F : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and a number $r_0 > 0$ such that

A)$F(\sigma_0) = 0$,

B)$D_F(\sigma_0) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is onto

C) for any $r, 0 < r < r_0$ we have $\Sigma \cap B_{\sigma_0}(r) = \{u \in B_{\sigma_0} | F(u) = 0\}$. [In other words $\Sigma \cap B_{\sigma_0}(r) = F^{-1}(0) \cap B_{\sigma_0}(r)$] where $B_{\sigma_0}(r) := \{u \in U | \|u - \sigma_0\| < r\}$.

$Q3$. There exists an open set $U \subset \mathbb{R}$, a smooth function $\phi : U \to \mathbb{R}^n$ and a number $r_0 > 0$ such that

A)$\phi(0) = \sigma_0$,

B)The linear map $D_\phi(0) : \mathbb{R} \to \mathbb{R}^n$ is an imbedding

C) for any $r, 0 < r < r_0$ we have $\Sigma \cap B_{\sigma_0}(r) = \phi(U) \cap B_{\sigma_0}(r)$ where $\phi(U) := \{\phi(x)\}, x \in (U)$ and

D)for any open set $U' \subset U$ we can find $r_0' > 0$ such that for any $r, 0 < r < r_0'$ such that we have $\Sigma \cap B_{\sigma_0}(r) = \phi(U') \cap B_{\sigma_0}(r)$

b)Let $C \subset \mathbb{R}^n$ be a *smooth curve* in $\mathbb{R}^n$ at $c \in \Sigma$. Give three definitions of the *tangent line* l_c and *tangent subspace* $T_\Sigma(c) \subset \mathbb{R}^n$ corresponding to the three definitions ($Q1, Q2$ and $Q3$) of the *smoothness*. That is assume that (C, c) satisfies $Qi, 1 \leq i \leq 3$ and give a definition of l_c and $T_\Sigma(c)$ in terms of the definition Qi.

A Solution. You define $T_\Sigma(c) \subset \mathbb{R}^n$ by

1) $T_\Sigma(c) = \{(f_1'(x_0^i)t, ..., f_{i-1}'(x_0^i)t, t, f_{i+1}'(x_0^i)t, ..., f_n'(x_0^i)t)\}, t \in \mathbb{R}$

2)$T_\Sigma(c) = Ker D_F(\sigma_0)$

3)$T_\Sigma(c) = ImD_\phi(0)$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such. We say that $c \in \Sigma$ is a point of *local minimum* of the restriction of f on C if there exists an open set $U \subset \mathbb{R}^n, u \in U$ such that for all $c' \in U \cap C$ we have $f(c') \leq f(c)$

c)Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $c \in \Sigma$ is a point of local minimum of the restriction of f on C. Show that the restriction of the differential $D_f(c)$ to the tangent subspace $T_C(c)$ is equal to zero.

Solution. Use the definition $Q3$ and the chain rule.

d) Let $C := S^2 \cap P$ where $S^2 := (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1, P := (x, y, z) \in \mathbb{R}^3 | x + y + z = 1/2$. Show that C is a smooth curve and find the point $c \in C$ such that $f(c) < f(c')$ for all $c' \in C$ where $f(x, y, z) := z$.

Solution.Consider the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by $F(x, y, z) := (x^2 + y^2 + z^2 - 1, x + y + z - 1/2)$. Then $C = F^{-1}(0)$. To show that C is smooth we have to check that for any $u = (x, y, z) \in C$ the linear map $D_F(u)\mathbb{R}^3 \to \mathbb{R}^2$ is onto. By the definition we have $D_F(u)(a, b, c) = (2xa + 2by + 2cz, a + b + c)$. It is clear that the map $D_F(u)$ is NOT onto only $x = y = z$. But C does not contain any points of the form (x, x, x). So we see that C is a smooth curve.

As follows from c) if $u = (x, y, z) \in C$ is a point of local minimum the restriction of $D_f(u)$ on $T_C(c)$ is equal to zero. In other words for any (a, b, c) such that $2xa + 2by + 2cz = 0$ and $a + b + c = 0$ we have $c = 0$. But this is true only if $x = y$. So we have to solve the system of equation

$$2x + z = 1/2$$

$$2x^2 + z^2 = 1$$

.

It has two solutions -one gives the minimum and the other the maximum of f on C.

4. Let A be a symmetric $n \times n$ matrix, $S^{n-1} := \{\bar{x} \in \mathbb{R}^n | < \bar{x}, \bar{x} >= 1\}$.

a) Show that for any $s \in S^{n-1}$ we have $T_{S^{n-1}}(s) = \{v \in \mathbb{R}^n | < v, s >= 0\}$

Solution. use the defintion $Q2$

Let $f_A : \mathbb{R}^n \to \mathbb{R}$ be a function such that $f_A(v) :=< Av, v >$.

b)Show that the restriction of the function f_A on S^{n-1} reaches it's minimum at some point $s \in S^{n-1}$.

Solution. Show that S^{n-1} is closed and bounded and that the function f_A is continuous.

c)Let $s \in S^{n-1} \subset \mathbb{R}^n$ be a point of maximum for the restriction of the function f_A on S^{n-1}. Show there exists a real number λ such that $As = \lambda s$.

We start with the following tow results.

Let v be a vector in $\mathbb{R}^n, v \neq 0$We denote by $L_v \subset \mathbb{R}^n$ the hyperplane $L_v = \{x \in \mathbb{R}^n | < v, x >= 0\}$.

Lemma 1. Let $w \in \mathbb{R}^n$ be a vector such that $< w, x >= 0$ for all $x \in L_v$. Then there exists $c \in \mathbb{R}$ such that $w = cv$.

Lemma 2. For any $v, w \in \mathbb{R}^n$ we have $D_{f_A}(v)(w) =< Av, w >$

I'll leave for you the proof of the Lemmas.

Proof of c)

Let $s \in S^{n-1} \subset \mathbb{R}^n$ be a point of maximum for the restriction of the function f_A on S^{n-1}. By 4a) and 3c) we see (?) that $< As, x >= 0$ for all $x \in L_s$.Therefore it follows from Lemmas that there exists $c \in \mathbb{R}$ such that $As = cs$.

d)Prove the existence of a basis $v_1, ..v_n \in \mathbb{R}^n$ such that $Av_i = \lambda_i v_i$ where $\lambda_i, 1 \leq i \leq n$ are real numbers.

A solution. Let $s_1 \in S^{n-1} \subset \mathbb{R}^n$ be a point of maximum for the restriction of the function f_A on S^{n-1}. Since $As_1 = c_1 s_1$ one can show (?) that the operator A preserves the subspace L_{s_1} [that is $Ax \in L_{s_1}$ for all $x \in L_{s_1}$. Now let $s_2 \in S^{n-1} \cap L_{s_1} \subset \mathbb{R}^n$ be a point of maximum for the restriction of the function f_A on $S^{n-1} \cap L_{s_1}$. the same argument show that $As_2 = c_2 s_2$ for some $c_2 \in \mathbb{R}$. E.t.c