MATH 23b, SPRING 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam (take-home portion)
May 12, 2002

Directions: You have until noon on Monday, May 20, to complete this exam, by which time it should be turned in to my mailbox in the Science Center. You may use your own class notes, your own homework assignments, and any published books as your only aids. You may not use any internet resources except for the course website and the posted homework solutions. You may not discuss the exam with anyone, and all questions should be directed only to the instructor. (In particular, please do not direct questions to the Math 23b CA's.) Please note that I will hold office hours on Friday afternoon, but only until 4 P. M.

There is partial credit, but only for intelligible work. Please write neatly, and please turn in clean copies of solutions, not random scribblings that may or may not have anything to do with a final answer. In fact, one point per problem will be awarded for *neatness only*, and one point will be awarded for *style only.* Make sure your name is prominently displayed on your work, and *please* staple your final pages together into one stack.

You may quote results from class and/or your notes with an appropriate reference, and you must cite anything you take from a published book. Otherwise, all work should be your own.

There are two problems on this exam. In the first (10 points), you are asked to show that increasing functions are integrable on closed intervals. The second problem has four parts, one of which is a reading assignment. In the first part (10 points), you will consider the problem of "differentiating under the integral" and show that a function defined by an integral may be so differentiated under certain hypotheses. In the last two parts (2 points each), you will use a generalization of this technique and a simple differential equation in order to find the explicit form of a function so defined.

1. Recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *increasing* if $f(x) < f(y)$ whenever $x < y$.

   Show that if $f : [a, b] \longrightarrow \mathbb{R}$ is increasing, then f is integrable on $[a, b]$.

   (Hint: In particular, f is defined on $[a, b]$.)

2. **Leibnitz' Rule** (from Spivak's *Calculus on Manifolds*, p. 62, #3-32)

   (a) Let $f : [a, b] \times [c, d] \longrightarrow \mathbb{R}$ be continuous, and suppose that $D_2 f$, the partial derivative of f with respect to its second variable, is continuous. Define $F : [c, d] \longrightarrow \mathbb{R}$ by the rule:

   $$F(y) = \int_a^b f(x, y)\, dx.$$

   Show that $F'(y) = \displaystyle\int_a^b D_2 f(x, y)\, dx$.

   (Hint: Use the Fundamental Theorem of single-variable Calculus to write $F(y) = \displaystyle\int_a^b \left(\int_c^y D_2 f(x, y)\, dy + f(x, c) \right) dx$.)

   (b) *(not required to be turned in)*

   Note that the result of part (a) is true in much greater generality. Compare with Edwards' problem #IV.3.5 from p. 233, in which the interval $[a, b]$ is replaced by any set that has positive content and the interval $[c, d]$ is replaced by an open interval, at a cost of insisting that $D_2 f$ now be *uniformly* continuous.

   For the next two parts, consider $F(y) = \displaystyle\int_0^\infty e^{-x^2} \cos 2xy\, dx$. (from Apostol's *Mathematical Analysis*, p. 302, #10.22)

   (c) Show that F satisfies the differential equation:

   $$F'(y) + 2y \cdot F(y) = 0.$$

   (Hint: You may assume that the result from part (a) applies even though F is defined by an improper integral.)

   (d) Solve the differential equation to conclude that $F(y) = \frac{1}{2}\sqrt{\pi} e^{-y^2}$.

   (Hints: 1. This is a straight-forward differential equation that is first-order and separable, and 2. You may assume without proof the result from class that $\displaystyle\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$.)