MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Solutions to Final Exam (take-home portion)

For the entirety of this (take-home) part of the exam, we consider the vector space $V = \mathbb{R}^n$ with the standard basis and the usual inner product. That is, if $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$, then $\langle \mathbf{v}, \mathbf{w} \rangle = v_1w_1 + \cdots + v_nw_n$. We define the collection of linear transformations (and their matrices with respect to the standard basis) from V to V to be:

$$M_n(\mathbb{R}) = \{A : V \longrightarrow V |\ A \text{ is linear}\}.$$

Throughout, we assume that $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ has the structure of Euclidean space. We also define the collection (which has the algebraic structure of a *group*) of invertible linear transformations, called the *general linear group*, to be:

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) |\ A \text{ is invertible}\}.$$

Finally, we define three special subgroups of the general linear group, called the *special linear group*, the *orthogonal group*, and the *special orthogonal group*, respectively, as follows:

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) |\ det(A) = 1\}$$

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) |\ A^tA = I\}$$

$$SO_n(\mathbb{R}) = SL_n(\mathbb{R}) \cap O_n(\mathbb{R}),$$

where A^t is the transpose of the matrix A (see HW #8.6) and I is the identity linear transformation.

1. **Some Basics.**

   (a) **Show that $det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous.**
   (Hint #1: Use the theorems, not the definition.
   Hint #2: This part is important for the rest of the exam.)

If we write a matrix $A \in M_n(\mathbb{R})$ in terms of its entries $\{a_{ij}\}_{i,j=1}^n$, then we see from its definition that $det(A)$ is a *polynomial* in the n^2 variables that are the entries.

From theorems proved in class, we know that constant functions are continuous and that "projection" functions (such as $f(x_1, \ldots, x_k) = x_i$) are continuous. We also know that products and sums of continuous functions are continuous, and a polynomial (like the determinant) is just such a sum of products of constant and projection functions.

(b) Determine which of these four groups (that is, not including $M_n(\mathbb{R})$) are subsets of the others on the list, and for all pairs G and H, give examples of 2×2 matrices A such that $A \in G$ but $A \notin H$ and vice versa, if possible. (Note that if you decide that $H \subset G$, then obviously you cannot find a matrix A such that $A \in H$ but $A \notin G$.)

Clearly, all three of the other groups are by definition subgroups of $GL_n(\mathbb{R})$. It is also clear from the definition that $SO_n(\mathbb{R})$ is a subgroup of both $SL_n(\mathbb{R})$ and $O_n(\mathbb{R})$. So that is five of the twelve comparisons. For the other seven, we provide the following examples when $n = 2$:

- $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is in $GL_2(\mathbb{R})$ but not in any of the other three.
- $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is in $O_2(\mathbb{R})$ but not in $SL_2(\mathbb{R})$ or $SO_2(\mathbb{R})$.
- $C = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right]$ is in $SL_2(\mathbb{R})$ but not in $O_2(\mathbb{R})$ or $SO_2(\mathbb{R})$.

2. Topology.

(a) Show that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$. (Hint: Use #1(a).)

The set $S = (-\infty, 0) \cup (0, \infty)$ is open in $\mathbb{R}$ and since $GL_n(\mathbb{R})$ is the inverse image of S under the determinant map (which is continuous), it too is open (from our major theorem about continuous functions).

(b) Show that $SL_n(\mathbb{R})$ is closed but not compact.

The set $S = \{1\}$ is closed in $\mathbb{R}$ and since $SL_n(\mathbb{R})$ is the inverse image of S under the determinant map (which is continuous), it too is closed (from our major theorem about continuous functions).

On the other hand, $SL_n(\mathbb{R})$ is not bounded (and hence not compact by the Heine-Borel Theorem) because, for any

$m \in \mathbb{N}$, the matrix $A = \begin{bmatrix} m & 0 & 0 & \cdots & 0 \\ 0 & 1/m & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$ is in $SL_n(\mathbb{R})$

and has norm $||A|| = \sqrt{m^2 + \frac{1}{m^2}} > m$.

(c) Show that the closure of $GL_n(\mathbb{R})$ is all of $M_n(\mathbb{R})$.

Let $A \in M_n(\mathbb{R})$. If $A \in GL_n(\mathbb{R})$, then A is already a limit point and we are done. Now suppose $A \notin GL_n(\mathbb{R})$. Then $det(A) = 0$ and A is not invertible. Another way to say this is that its columns are linearly dependent. Suppose that $dim(span\{A\mathbf{e}_i\}) = k < n$.

Given $\varepsilon > 0$, we will show that there is a matrix $B \in M_n(\mathbb{R})$ such that $||B - A|| < \frac{\varepsilon}{n}$ and $dim(span\{B\mathbf{e}_i\}) = k + 1$. Then $n - k$ applications of this result will produce a matrix $Z \in M_n(\mathbb{R})$ such that $dim(span\{Z\mathbf{e}_i\}) = n$, and the triangle inequality will assure us that $||A - Z|| < \varepsilon$. Of course, if the columns of Z span an n-dimensional space, then Z is invertible, so in fact, $Z \in GL_n(\mathbb{R})$.

Let $\varepsilon > 0$ and suppose that $V = span\{A\mathbf{e}_i\}$, a subspace of $\mathbb{R}^n$. We suppose, as noted above, that $dim(V) = k$. Take k of the columns from A that span V, and let $\mathbf{v}$ be any of the *other* columns of A (which will automatically be a linear combination of the first k, by our assumptions). Now let $\mathbf{w}$ be *any* vector in $\mathbb{R}^n$ not in V. (We can do this much in the manner that we use to extend a basis.) Then $\mathbf{w}$ is not a linear combination of the basis for V, and hence neither is $\mathbf{v} + c\mathbf{w}$ for any real number c. In particular, we choose $c = \frac{\varepsilon}{2n||\mathbf{w}||}$. Finally, we construct B from A by replacing the column $\mathbf{v}$ with $\mathbf{v} + c\mathbf{w}$. It follows that $||B - A|| = ||c\mathbf{w}|| = \frac{\varepsilon}{2n} < \frac{\varepsilon}{n}$, and the fact that $dim(span\{B\mathbf{e}_i\}) = k + 1$ follows from the linear independence condition.

3. Orthogonal Matrices.

(a) Show that $A \in O_n(\mathbb{R})$ if and only if A is inner-product preserving. (See HW #8.5. Hint: The inner-product may be written as follows: $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^t \mathbf{w}$, where v and w are thought of as $n \times 1$ matrices, in coordinates with respect to the standard basis.)

($\Rightarrow$): We follow the hint and suppose that A is orthogonal. If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then:

$$\begin{aligned}
\langle A\mathbf{v}, A\mathbf{w} \rangle &= (A\mathbf{v})^t (A\mathbf{w}) && \text{by the hint} \\
&= \mathbf{v}^t A^t A \mathbf{w} && \text{by properties of transpose} \\
&= \mathbf{v}^t I \mathbf{w} && \text{since } A \text{ is orthogonal} \\
&= \mathbf{v}^t \mathbf{w} && \text{by properties of } I \\
&= \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

so A is inner-product preserving.

($\Leftarrow$): Choose $\mathbf{e}_i$ and $\mathbf{e}_j$ to be two of the standard basis vectors. If A is inner-product preserving, then:

$$\begin{aligned}
\langle A\mathbf{e}_i, A\mathbf{e}_i \rangle &= \langle \mathbf{e}_i, \mathbf{e}_i \rangle = 1 \\
\langle A\mathbf{e}_i, A\mathbf{e}_j \rangle &= \langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0 \quad \text{if } i \neq j
\end{aligned}$$

and hence the columns of A (given by $A\mathbf{e}_i$) are mutually orthogonal and of norm 1, so by problem #3(b) below, A is orthogonal.

(b) Show that the columns of any $A \in O_n(\mathbb{R})$ form an orthonormal basis for V by showing that:

i. The columns of any $A \in O_n(\mathbb{R})$ are vectors of norm 1.

ii. The columns of any $A \in O_n(\mathbb{R})$ are mutually orthogonal.

We will show more than is asked! Namely, that the columns of A form an orthonormal basis for V if and only if $A \in O_n(\mathbb{R})$.

(If): Suppose A is orthogonal. Then $A^t A = I$.

If $A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} = \begin{bmatrix} | & & | \\ A\mathbf{e}_1 & \cdots & A\mathbf{e}_n \\ | & & | \end{bmatrix}$,

then $A^t = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{n1} \\ \vdots & & \vdots \\ \alpha_{1n} & \cdots & \alpha_{nn} \end{bmatrix}$, so the ij-th entry of $A^t A$ is the dot product (in $\mathbb{R}^n$) of the i-th row of A^t with the j-th column of A, which are $A\mathbf{e}_i$ and $A\mathbf{e}_j$, respectively.

Since $A^tA = I$, we see that any diagonal entry of A^tA is 1, and hence $||A\mathbf{e}_i|| = \sqrt{\langle A\mathbf{e}_i, A\mathbf{e}_j\rangle} = 1$, for every i. (This says that the columns of A have norm 1.)
Also since $A^tA = I$, we see that any off-diagonal entry of A^tA is 0, and hence $\langle A\mathbf{e}_i, A\mathbf{e}_j\rangle = 0$ for $i \neq j$. (This says that the columns of A are mutually orthogonal.)

(Only If): Conversely, suppose the columns of A have norm 1 and are mutually orthogonal. Then the ij-th entry of A^tA is either 1 (if $i = j$) or 0 (if $i \neq j$), by the argument presented above. In short, $A^tA = I$, and hence $A \in O_n(\mathbb{R})$.

4. The group $O_n(\mathbb{R})$.

(a) If $A \in O_n(\mathbb{R})$, find all possible values of $det(A)$.

Assume A is orthogonal. We use the fact that $det(A^t) = det(A)$ (see Shilov p. 8) and that $det(AB) = det(A)det(B)$ to write:

$$1 = det(I) = det(A^tA) = det(A^t)det(A) = det(A)^2,$$

from which we immediately conclude that $det(A) = \pm 1$.

(b) Show that $O_n(\mathbb{R})$ is compact in $M_n(\mathbb{R})$ by showing that:

i. $O_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.

Consider the function $f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ given by $f(A) = A^tA$. This function is continuous because each of the entries of the resulting matrix is a polynomial involving the entries of the original matrix. Now we consider the single point $\{I\} \subset M_n(\mathbb{R})$, which is closed. Then $O_n(\mathbb{R}) = f^{-1}(\{I\})$, and since f is continuous, this set (the orthogonal group) is also closed.

ii. $O_n(\mathbb{R})$ is bounded in $M_n(\mathbb{R})$.
(Hint: Consider problem #3, especially part (b).)

Writing out the norm of a matrix A, considered as an element of $\mathbb{R}^{n^2}$, we see that

$$||A|| = \sqrt{||A\mathbf{e}_1||^2 + \cdots + ||A\mathbf{e}_n||^2},$$

where the norms under the square root are the norms of vectors in $\mathbb{R}^n$. The vectors $\{A\mathbf{e}_i\}$ are the columns of A and since A is orthogonal, we know from #3(b) that these columns have norm 1, and so, for any A, we apply the above formula to conclude that $||A|| = \sqrt{n}$. Hence $O_n(\mathbb{R})$ is bounded by the positive real number $\sqrt{n}$.

5. Permutation Matrices.

Let $\sigma \in S_n$ be a permutation, and define $A_\sigma \in GL_n(\mathbb{R})$ to be the linear transformation given by $A_\sigma \mathbf{e}_i = \mathbf{e}_{\sigma(i)}$ for the standard basis vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and extended by linearity.

(a) Determine $det(A_\sigma)$ for each $\sigma \in S_n$.

The best result is that $det(A_\sigma) = sgn(\sigma)$, where sgn is the signum of the permutation. This formula comes as the result of expanding the determinant. In general,

$$det(A) = \sum_{\tau \in S_n} (sgn\ \tau) \cdot \alpha_{1\tau(1)} \cdots \alpha_{n\tau(n)}.$$

When this is written out with $A = A_\sigma$, the only non-zero term in the sum appears when $\tau = \sigma^{-1}$, in which case, all of the corresponding α's are equal to 1, and hence $det(A_\sigma) = sgn(\sigma^{-1}) = sgn(\sigma)$.

(b) Show that $A_\sigma \in O_n(\mathbb{R})$ for each $\sigma \in S_n$.

We study the columns of A_σ, and use #3(b) (in which we proved the statement to be necessary and sufficient!) to show that since they form an orthonormal basis for $\mathbb{R}^n$, A_σ is orthogonal.
As for these columns having norm 1, we check that

$$||A_\sigma \mathbf{e}_i|| = ||\mathbf{e}_{\sigma(i)}|| = 1.$$

For mutual orthogonality, we suppose that $i \neq j$ and note that since σ is a bijection, we get

$$\langle A_\sigma \mathbf{e}_i, A_\sigma \mathbf{e}_j \rangle = \langle \mathbf{e}_{\sigma(i)}, \mathbf{e}_{\sigma(j)} \rangle = 0.$$

6. **Bonus.**

(a) **(Easy) Show that $O_n(\mathbb{R})$ is disconnected as a subset of $M_n(\mathbb{R})$.**

Let $B_1 = \{A \in M_n(\mathbb{R}) |\ det(A) > 0\}$ and let $B_2 = \{A \in M_n(\mathbb{R}) |\ det(A) < 0\}$. These are two disjoint open sets which cover $O_n(\mathbb{R})$ non-trivially (as we demonstrate below) and hence $O_n(\mathbb{R})$ is disconnected.

As for the sets being open, we note that $B_1 = (det)^{-1}(0, +\infty)$, and so is the inverse image of an open set under a continuous map. Similarly, B_2 is open. Their empty intersection follows immediately from their definitions, and the fact that each covers part of $O_n(\mathbb{R})$ is because $I \in B_1$ and the matrix $\begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is in B_2 and both are orthogonal.

(b) **(Hard) Show that $SO_n(\mathbb{R})$ is connected.**

In the case $n = 2$, we show that $SO_2(\mathbb{R})$ is path-connected and hence connected. It may easily be verified from its definition that

$$SO_2(\mathbb{R}) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \middle|\ \theta \in [0, 2\pi) \right\}.$$

Take $A, B \in SO_2(\mathbb{R})$ with corresponding angles θ_0 and θ_1 (and suppose without loss of generality that $\theta_0 < \theta_1$).

Define the map $f : [0, 1] \longrightarrow SO_2(\mathbb{R})$ by

$$f(x) = \begin{bmatrix} \cos(x\theta_1+(1-x)\theta_0) & -\sin(x\theta_1+(1-x)\theta_0) \\ \sin(x\theta_1+(1-x)\theta_0) & \cos(x\theta_1+(1-x)\theta_0) \end{bmatrix}.$$

It is clear that f is continuous (it is a composition of continuous functions) and that $f(0) = A$ and $f(1) = B$, and hence $SO_2(\mathbb{R})$ is path-connected, and so by the theorem, it is also connected.

For $n > 2$, the proof is still hard and uses induction.