MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 10
Due: December 7, 2001

Please turn in five separate sets labelled A through E.

1. Re-read Sections 1.8, 2.1 and the Appendix from Edwards.

2. (A) Let V and W be normed vector spaces, and let $f : V \longrightarrow W$ be a continous function. Show that if $S \subset V$ is compact, then $f(S) \subset W$ is compact.

   (Edwards proves this as Theorem 8.7 using his definition of compactness in terms of limit points. For this exercise, I would like you to prove it using our definition of compactness in terms of open covers.)

3. (B) In the p-adic field $\mathbb{Q}_p$, we define the ring of p-adic integers $\mathbb{Z}_p$ to be the analytic completion of $\mathbb{Z}$ with respect to the p-adic norm, that is, the equivalence classes of Cauchy sequences of integers. In other words, we define $\mathbb{Z}_p = \overline{\mathbb{Z}}$, where the closure is given in the p-adic norm.

   (a) Show that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p| \; |x|_p \leq 1\}$.
   (This is mostly a matter of verifying one of the assertions from HW #8.10, namely, that if $\{a_n\}$ is a Cauchy sequence converging to the p-adic number x, then the definition that $|x|_p = \lim_{n\to\infty} |a_n|_p$ does not depend on the equivalence class of $\{a_n\}$, i. e. is well-defined.)

   (b) Show that $\mathbb{Z}_p$ is both open and closed.

4. (A) In class, we proved that if $\{Q_n\}$ is a nested sequence of non-empty compact sets, then $\bigcap Q_n$ is compact and non-empty. (Recall that the sequence is nested if $Q_{n+1} \subset Q_n, \forall n$.)

   Find an example of a nested sequence of non-empty closed sets whose intersection is empty. (See Edwards, problem 8.10.)

5. (C) Let V and W be normed vector spaces, and let $f : V \longrightarrow W$ be linear. Show that f is continuous if and only if it is continuous at $\mathbf{0}$.

6. (C) Given an open set $S \subset \mathbb{R}^n$ and a point $\mathbf{x} \in S$, find a ball Q with rational center (that is, whose coordinates are rational) and rational radius such that $\mathbf{x} \in Q \subset S$.

7. (D) Let $B = \overline{B_1(\mathbf{0})} \setminus \{(0, 1)\} \subset \mathbb{R}^2$ be the closed unit ball with one point deleted. Construct an open cover of B with no finite subcover.

8. (D) Lemma 8.4 in Edwards states that if $A \subset \mathbb{R}^n$ is compact and $B \subset A$ is closed, then B is compact. Prove this using our definition of compactness (that is, not using Edwards' definition and not using the theorem that closed and bounded is equivalent to compact for subsets of $\mathbb{R}^n$).

9. (E) Let

$$\ell^\infty = \left\{ (a_1, a_2, a_3, \ldots) \middle| a_n \in \mathbb{R}, \forall n \quad \text{and } \{|a_n| \; |n \in \mathbb{N}\} \text{ is bounded} \right\}$$

be a subspace of the vector space of all sequences of real numbers, and consider the norm given by:

$$||\{a_n\}|| = \sup\{|a_n| \; |n \in \mathbb{N}\},$$

where the supremum ("sup," pronounced like "soup") of a bounded set of real numbers is the least upper bound.

   (a) Show that this norm satisfies the triangle inequality.

   (b) Let $\mathbf{v}_n \in \ell^\infty$ be the vector with all coordinates equal to zero except for a 1 in the n-th position. Show that $\{\mathbf{v}_n | n \in \mathbb{N}\}$ is an infinite, bounded, closed, and discrete set. (Comment: See HW #9.10(b). This shows that we also need the hypothesis that the space be finite-dimensional in order to draw our conclusion.)