MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 2
Due: September 28, 2001

Beginning with this problem set, we ask that you turn in five separate sheets (or sets of sheets) labelled A through E, so that the CA's may grade them in parallel. The individual problems are each labelled with one of A through E below.

1. Read Sections 1.1–1.5 of Edwards, and Sections 2.1–2.4 of Shilov.

2. (A) Show that $\mathbb{N}$ is unbounded (above) as a subset of $\mathbb{R}$.

   (Recall that the natural numbers $\mathbb{N}$ satisfy the Peano axioms. The important one as far as this problem is concerned is that if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$.)

3. (B) If $V$ is a vector space and $\mathbf{v} \in V$, show that $(-1)\mathbf{v} = -\mathbf{v}$.

   (Note that there really is something to do here. The expression on the right is the additive inverse of the vector, and the one on the left is the vector multiplied by the scalar $-1$, and these are not *a priori* the same thing.)

4. (A) Show that the set of ordered triples $(x, y, z)$ of real numbers satisfying the equations $2x + y = 0$ and $3y - z = 0$ form a vector space over the field $\mathbb{R}$.

5. (B) Recall the vector space $C[0,1] = \{f : [0,1] \to \mathbb{R} | f \text{ is continuous}\}$. Now, for each $c \in \mathbb{R}$, consider the set of vectors defined by

$$V(c) = \{f : [0,1] \to \mathbb{R} | f \text{ is continuous, and } f(0) = c\}.$$

   For which values of $c$ is $V(c)$ a subspace of $C[0,1]$? Explain.

6. (C) For $k = 1, 2$, recall that we have defined the two spaces:

$$\ell^k = \left\{ (a_0, a_1, a_2, \ldots) \middle| a_i \in \mathbb{R}, \forall i \geq 0, \text{and } \sum_{n=0}^{\infty} |a_n|^k < \infty \right\}$$

   (a) Show that $\ell^2$ is a vector space by showing that the addition of vectors and scalar multiplication are closed operations. That is, if $\mathbf{u}, \mathbf{v} \in \ell^2$ and $a \in \mathbb{R}$, then $\mathbf{u} + \mathbf{v} \in \ell^2$ and $a\mathbf{u} \in \ell^2$. (You need not verify that axioms V1-V8 are satisfied, as these are straightforward.)

   (b) Show that $\ell^1 \subset \ell^2$, and that $\ell^1 \neq \ell^2$.

7. (D) Give a reasonable definition of multiplication of vectors in $\mathbb{R}^2$, and discuss why this definition is unsatisfactory. There is obviously no single right answer here, but a good answer should discuss the axioms that we usually desire for our binary operations.

8. (D) *Read Edwards, problem # 1.6.*

   Let $S$ be some set with finite cardinality $n$. (For simplicity, you may assume that $S = \{1, 2, \ldots, n\}$.) Find a basis for the vector space of functions $V = \{f : S \to \mathbb{R}\}$, where addition and scalar multiplication are defined as for functions of one real variable.

9. (E) Find an infinite set of linearly independent vectors in $C[0, 1]$, not including any polynomials.

10. (E) Consider the field $F = \mathbb{Z}/7\mathbb{Z}$. Recall that we may think of the elements of $F$ as equivalence classes of integers, where $a \sim b$ if and only if $a - b$ is evenly divisible by 7, and where addition and multiplication are derived from $\mathbb{Z}$. (For example, $[4] \cdot [5] = [6]$ because $4 \cdot 5 = 20$, which has remainder 6 when divided by 7.)

    For simplicity, we often drop the equivalence class notation and denote this field by $\mathbb{Z}/7\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6\}$.

    Let $\mathbf{u} = (1, 0, 6)$ $\mathbf{v} = (1, 2, 1)$, and $\mathbf{w} = (2, 1, 3)$ be three vectors in $F^3$, that is, the set of ordered triples with coordinates in $F$. Find coefficients $a, b, c \in F$ to express the vector $\mathbf{x} = (1, 2, 3)$ as a linear combination $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

11. (*Not required*) Consider the spaces $\ell^1$ and $\ell^2$ as defined in Problem # 6, but with the terms $a_i$ taken from the finite field $\mathbb{Z}/2\mathbb{Z}$. Give an explicit condition for deciding whether of not a particular sequence is in $\ell^1$ or $\ell^2$. (What must happen for such an infinite series to converge? Recall that the limits and sums are being taken in the finite field!)