MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 3
Due: October 5, 2001

Continuing with this problem set, we ask that you turn in five separate sheets (or sets of sheets) labelled A through E, so that the CA's may grade them in parallel. The individual problems are each labelled with one of A through E below.

1. Read Sections 1.4–1.5 of Edwards, and Sections 4.1–4.3 and 4.6 of Shilov.

2. (A) Let $V = \{(a_0, a_1, a_2, \ldots)|a_i \in \mathbb{R}\}$ be the vector space of all infinite sequences of real numbers. Let W be the subspace of V consisting of all *arithmetic* sequences. Find a basis for W, and determine the dimension of W. (*A sequence is arithmetic is there is some constant c such that $a_{n+1} - a_n = c$ for all $n \geq 0$.*)

3. (A) In the following, we consider the *shift* operator.

   (a) Consider the linear map $S : \mathbb{R}^3 \to \mathbb{R}^3$ which acts as follows: $S(x, y, z) = (0, x, y)$. Find the kernel and image of S, and verify that $dim(Ker(S)) + dim(Im(S)) = dim(\mathbb{R}^3)$.

   (b) Now consider the linear map $S : V \to V$, where V is the vector space of all infinite sequences of real numbers as in Problem #1, and where L acts as follows: $S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$. Find the kernel and image of S. How does the result about the dimensions of kernels and images apply?

4. (B) Let $P_n = \{p(x) = a_0 + a_1x + \cdots + a_nx^n|a_i \in \mathbb{R}, \forall i\}$ be the vector space of all polynomials of degree less than or equal to n. Consider the map $L : P_n \to \mathbb{R}$ defined by $L(p) = \int_0^1 p(x)\, dx$.

   (a) Show that L is a linear map.

   (b) Determine $Im(L)$, and find a basis.

   (c) Use the isomorphism $P_n/Ker(L) \cong Im(L)$ or the fact that $dim(Ker(L)){+}dim(Im(L)) = dim(P_n)$ to help determine $dim(Ker(L))$, and find a basis for $Ker(L)$.

5. (C) Let V and W be subspaces of some finite-dimensional vector space U. Show that
$$V/(V \cap W) \cong (V + W)/W.$$

6. (B) Show that if V is a vector space with $dim(V) = n$, then any collection of $n + 1$ vectors in V is linearly dependent.

7. (D) How many vectors are in the vector space $V = (\mathbb{Z}/p\mathbb{Z})^n$, where p is a fixed prime number? How many distinct one-dimensional subspaces does V have? How many distinct two-dimensional subspaces does V have?

8. (D) Suppose U is a finite-dimensional vector space and V is a subspace of U. Show that $dim(U/V) = dim(U) - dim(V)$.

9. (E) Let P_n be the vector space of polynomials (with real coefficients) of degree less than or equal to n, and let P_n^0 be the subspace of polynomials with terms of only even degree. Find a basis for P_n/P_n^0.

10. (E) Find the kernel of the linear map $L : (\mathbb{Z}/7\mathbb{Z})^3 \longrightarrow (\mathbb{Z}/7\mathbb{Z})^3$ given by $L(x, y, z) = (x + 2z, 2x + 3y + 4z, 4x + 3y + z)$.