MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 4
Due: October 12, 2001

Please turn in four separate sets labelled A through D.

1. Read Sections 4.3–4.7 of Shilov, especially subsection 4.68.
2. (A) **The Second Isomorphism Theorem**

   Suppose V and W are subspaces of a vector space U.

   Then $V/(V \cap W) \cong (V + W)/W$.

   (a) Prove the theorem, assuming $dim(U) < \infty$. (Hint: Construct and extend bases for the various spaces.)

   (b) Prove the theorem without assuming that $dim(U) < \infty$.
   (Of course, if you do this correctly, there is no need to do part (a).)
3. (B) Recall the shift operator from problem # 3 last week. Let V be the vector space of all infinite sequences of real numbers, and define $S : V \longrightarrow V$ by the following: $S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$

   Now define $T : V \longrightarrow V$ by the following: $T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$

   (a) Show that $T \circ S = I$ but that $S \circ T \neq I$, where $I : V \longrightarrow V$ is the identity map.

   (b) Which of S and T is onto? Which is one-to-one? Explain.
4. (B) Find the kernel of the linear differential operator $D : C^\infty \longrightarrow C^\infty$ given by $D(f) = f' + af$, where a is some fixed real number.
5. (C) Let $A : U \rightarrow V$ and $B : V \rightarrow W$ be linear maps between finite-dimensional vector spaces.

   (a) Show that if A and B are both surjective, then so is $B \circ A$.

   (b) Show that if B is one-to-one, then $Ker(B \circ A) = Ker(A)$.

   (c) Assuming A and B are surjective, show that $dim(Ker(B \circ A)) = dim(Ker(A)) + dim(Ker(B))$.
6. (D) Find the inverse of the linear map $L : (\mathbb{Z}/7\mathbb{Z})^3 \longrightarrow (\mathbb{Z}/7\mathbb{Z})^3$ given by $L(x, y, z) = (x + y + z, 2x + 3y + 4z, 3x + 4y + 6z)$.
7. (D) Problem #5 from Shilov, Chapter 4, page 114.