MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 5
Due: October 19, 2001

Please turn in four separate sets labelled A through D.

1. Read Sections 2.4 and 4.8–4.9 of Shilov.
2. (A) Consider the linear differential operator $D : C^\infty \longrightarrow C^\infty$ given by $D(f) = f' + af$, where a is some fixed real number.
   (a) Find *all* real eigenvalues and eigenvectors for this operator.
   (b) Show that this operator is surjective. (Hint: Let $g \in C^\infty$, and show that $f' + af = g$ has a solution. Multiply both sides by the *integrating factor* e^{ax}, and integrate both sides–on the left using the product rule!)
3. (A) Let F be any field. Show that the linear operator $L : F^2 \longrightarrow F^2$ given by $L(x, y) = (ax+by, cx+dy)$ is invertible if and only if $ad-bc \neq 0$.
4. (B) Suppose λ is a non-zero eigenvalue for the linear transformation $A : V \longrightarrow V$.
   (a) Show that λ^2 is an eigenvalue for A^2.
   (b) If A is invertible, show that λ^{-1} is an eigenvalue for A^{-1}.
5. (B) Suppose $A : V \longrightarrow V$ is a linear map, and suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a set of non-zero eigenvectors for A with distinct eigenvalues $\lambda_1, \dots, \lambda_m$. Show that these vectors are linearly independent. (Hint: Use induction on m.)
6. (C) Let $V = (\mathbb{Z}/7\mathbb{Z})^3$, and consider the linear map $L : V \longrightarrow V$ given by $L(x, y, z) = (x + y + z, 2y + 3z, 4z)$. Find the eigenvalues of L, and find an eigenbasis for V. (Hint: Look for likely choices of eigenvalues–I claim that two of them are easy, and the third follows a pattern.)
7. (D) Find examples of *invertible* linear transformations $A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that:
   (a) A has no eigenvalues.
   (b) A has only one eigenvalue λ, but $dim(V_\lambda) < 4$.
   (c) $\mathbf{e}_1 = (1, 0, 0, 0)$ and $\mathbf{v} = (1, 1, 1, 1)$ are both eigenvectors but have distinct eigenvalues.

8. (C) Prove the following:

   Theorem: Let U and V be vector spaces over the same field.

   If $dim(U) = n$ and $dim(V) = m$, then $dim(U \oplus V) = n + m$.

9. (D) Two linear transformations $A : V \longrightarrow V$ and $B : V \longrightarrow V$ are said to be *similar* if there exists an invertible linear transformation $S : V \longrightarrow V$ such that $A = SBS^{-1}$. Consider the following, where a well-written answer may suffice for both parts:

   (a) Show that if A and B are similar, then they have the same spectra. (That is, if λ is an eigenvalue for A, then λ is an eigenvalue for B, and vice versa.)

   (b) Suppose A and B are similar and λ is an eigenvalue for both. Find the precise relationship between the eigenspace for λ with respect to A and the eigenspace for λ with respect to B.

10. (Not required)

    Show that if $A : U \longrightarrow V$ and $B : V \longrightarrow W$, then $(BA)^{-1} = A^{-1}B^{-1}$.