MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 6
Due: October 26, 2001

Please turn in five separate sets labelled A through E.

1. Read Sections 1.2, 2.3, and 3.1–3.3 of Shilov.

2. (A) Let P_n be the vector space of polynomials of degree less than or equal to n, with real coefficients. Let $D : P_n \longrightarrow P_n$ be the usual differentiation operator. Find non-trivial subspaces of P_n of each possible dimension which are invariant under D. (Note that the eigenspaces are insufficient to describe the behavior of D on P_n, because there is only one one-dimensional eigenspace. In other words, P_n is not diagonalizable with respect to D for $n \neq 0$, and so we consider its invariant subspaces.)

3. (A) Let P_n be as above. Define the following subspaces:

$$P_n^0 = \{p(x) \in P_n | p(-x) = p(x), \forall x\}$$

$$P_n^1 = \{p(x) \in P_n | p(-x) = -p(x), \forall x\}$$

(Note that the definition of P_n^0 is equivalent to the one we gave in HW #3.9. The first of these are called *even* polynomials and the second are the *odd* polynomials.)

   (a) Show that P_n^0 and P_n^1 are not invariant under D but are invariant under D^2.

   (b) Show that $P_n \cong P_n^0 \oplus P_n^1$. (If you use bases for the two subspaces for this argument, then you should show that they *are* bases.)

4. (B) Let $\ell = \{(a_0, a_1, a_2, \ldots) | a_i \in \mathbb{R}\}$ to be the vector space of all infinite sequences of real numbers. Define

$$\ell^0 = \{(a_0, a_1, a_2, \ldots) | a_i \in \mathbb{R}, \text{and } a_i = 0 \text{ for all but finitely many } i\}$$

(Note that the definition of ℓ^0 is consistent with the one we gave in HW #2.6 for ℓ^1 and ℓ^2.)

   (a) (Not required) Convince yourself that ℓ^0 is a subspace of ℓ.

   (b) Does ℓ^0 have a complement in ℓ? Explain.

   (c) Let $V = \text{span}\{(1, 1, 1, \ldots)\}$. Find a complement to V in ℓ.

5. (C) We define a linear map $P : V \longrightarrow V$ to be a *projection* provided that $P^2 = P$. In the following, consider such a projection P:

   (a) Show that 0 and 1 are the only possible eigenvalues of P.

   (b) If V_λ represents the eigenspace for the eigenvalue λ under the map P, show that $V \cong V_0 \oplus V_1$. (Note that this shows that a projection is diagonalizable!)

6. (D) Let P_3 be the vector space of polynomials of degree less than or equal to 3, with real coefficients.

   Let $\mathfrak{B}_1 = \{1, x, x^2, x^3\}$, $\mathfrak{B}_2 = \{1, 1 + x, 1 + x^2, 1 + x^3\}$, and $\mathfrak{B}_3 = \{1 + x, 1 - x, x^2 - x^3, x^2 + x^3\}$ be bases for P_3, and let $D : P_3 \longrightarrow P_3$ be the usual differentiation operator. Write down the matrices for D with respect to the three bases.

7. (E) Let $V = \mathbb{R}^2$ be two-dimensional Euclidean space, with its usual x- and y- coordinate axes. Consider the linear transformation $L_\alpha : V \longrightarrow V$ that performs a reflection about the line $y = \alpha x$.

   (a) Write the matrix for L_α with respect to the basis $\mathfrak{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$. (Hint: Use elementary geometry to compute $L_\alpha(\mathbf{e}_1)$ and $L_\alpha(\mathbf{e}_2)$.)

   (b) Calculate the matrix for $L_\beta \circ L_\alpha$ (with respect to $\mathfrak{B}$) in two ways: by multiplying the matrices for L_β and L_α, and by determining the matrix for the resulting composed linear transformation directly.

   (c) Show that the composed linear transformation $L_\beta \circ L_\alpha$ is a rotation. By what angle are vectors in $\mathbb{R}^2$ rotated under this transformation?

8. (C) *In the following, we will not distinguish notationally between a linear transformation and its matrix.*

   Let V be a finite-dimensional vector space, and let $A : V \longrightarrow V$ be linear.

   (a) Show that: $Im(A) \subset Ker(A)$ if and only if $A^2 = 0$.
   (Bonus: When is such an A diagonalizable?)

   (b) If $V = \mathbb{R}^2$, then find all such matrices A.

   (c) If $V = (\mathbb{Z}/2\mathbb{Z})^2$, then find all such matrices A.