MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 7
Due: November 9, 2001

Please turn in three separate problems labelled A through C.

1. Read Sections 1.3–1.9, 3.6, and 4.7 of Shilov, Section 1.6 of Edwards, and the hand-outs from Halmos.

2. (Not required) Let V be a vector space over F. Consider the k-linear forms $f : V^k \longrightarrow F$. For any two such forms f_1 and f_2 and any scalar $c \in F$, we define:

$$(f_1 + f_2)(\mathbf{v}_1, \mathbf{v}_2) = f_1(\mathbf{v}_1, \mathbf{v}_2) + f_2(\mathbf{v}_1, \mathbf{v}_2), \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in V$$

$$(cf_1)(\mathbf{v}_1, \mathbf{v}_2) = c \cdot f_1(\mathbf{v}_1, \mathbf{v}_2), \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in V$$

   Convince yourself that the collection of k-linear forms on V form a vector space over F with addition and scalar multiplication defined as above.

3. (A) Show that not every skew-symmetric multilinear form $f : V^n \longrightarrow F$ is alternating by constructing an example. (Note that the only cases where this can happen are over fields F wherein $1 + 1 = 0$. Follow Halmos' reasoning in the proof that alternating implies skew-symmetric, Section 30, Theorem 1.)

4. (B) Let $dim(V) = n$, and let $f : V^k \longrightarrow F$ be an alternating k-linear form with $k < n$. Show by example that it is possible to have a set of k linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in V such that $f(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0$. (Make sure that $k \geq 2$ so that f can be alternating!)

5. (C) In the following, we consider vector spaces of bilinear forms:

   (a) Let $V = \mathbb{R}^2$. Show that the form $f : V^2 \longrightarrow \mathbb{R}$ defined by $f((a, b), (c, d)) = ad - bc$ is bilinear and alternating.

   (b) Now let $V = \mathbb{R}^3$. Construct two *linearly independent* alternating bilinear forms $f : V^2 \longrightarrow \mathbb{R}$.

   (c) Determine the dimensions of the spaces of alternating bilinear forms on $V = \mathbb{R}^2$ and $V = \mathbb{R}^3$.