

MATH 23b, SPRING 2002  
THEORETICAL LINEAR ALGEBRA  
AND MULTIVARIABLE CALCULUS  
Homework Assignment # 7  
Due: April 12, 2002

Homework Assignment #7 (Final Version)

1. Read Edwards Sections 4.3 and 4.4.

Problems 2 and 3 refer to the following theorem that we proved in class:

**Theorem.** Let  $A \subset \mathbb{R}^n$  be a closed rectangle, and let  $f : A \rightarrow \mathbb{R}$ . Let  $B = \{x \in A \mid f \text{ is discontinuous at } x\}$ . Then  $f$  is integrable if and only if  $B$  has measure zero.

2. In the “if” part of the proof, we considered the set:

$$B^\varepsilon = \{x \in A \mid o(f, x) \geq \varepsilon\}$$

and claimed that it was closed (and hence compact since  $A$  is bounded). Prove this.

3. In the “only if” part of the proof, we chose a partition  $P$  of  $A$  such that  $U(f, P) - L(f, P) < \frac{\varepsilon}{n}$  and covered the set  $B^{\frac{1}{n}}$  with rectangles in the set

$$\mathcal{S} = \{S \in P \mid S \cap B^{\frac{1}{n}} \neq \emptyset\}.$$

We then claimed that for any  $S \in \mathcal{S}$ , we had  $M_S(f) - m_S(f) \geq \frac{1}{n}$ .

Alas, this last statement is not quite true, but that doesn't mean that we can't patch the proof in the following way.

- (a) Show by example (keep it simple!) that it is possible to have an  $S \in \mathcal{S}$  such that  $M_S(f) - m_S(f) = 0$ .
  - (b) Modify the choice of the collection  $\mathcal{S}$  so that the statement  $M_S(f) - m_S(f) \geq \frac{1}{n}$  is true for all  $S \in \mathcal{S}$ .
- (Hints)
- i. We only need to cover the set of points  $x \in A$  such that  $o(f, x) \geq \frac{1}{n}$ .
  - ii. The previous definition of  $\mathcal{S}$  is redundant. In other words, there is a subset of  $\mathcal{S}$  which covers  $B^{\frac{1}{n}}$  on which the statement is true.
  - iii. We might say that given  $a \in S$ , the oscillation  $o(f, a)$  is “realized” if there is a sequence of points  $\{x_i\}_{i=1}^\infty \subset S$  such that  $x_i \rightarrow a$  and  $\lim_{i \rightarrow \infty} |f(x_i) - f(a)| = o(f, a)$ .

4. Let  $A \subset \mathbb{R}^n$  be a closed rectangle, and let

$$I = \{f : A \longrightarrow \mathbb{R} \mid f \text{ is integrable on } A\}.$$

- (a) Show that  $I$  is a vector space over  $\mathbb{R}$  by showing that if  $f_1, f_2 \in I$  and  $c_1, c_2 \in \mathbb{R}$ , then  $c_1 f_1 + c_2 f_2 \in I$ . (In other words,  $I$  is a subspace of the vector space  $V = \{f : A \longrightarrow \mathbb{R}\}$ .)
- (b) Show that  $\int_A c f = c \int_A f$  and  $\int_A (f_1 + f_2) = \int_A f_1 + \int_A f_2$ .
5. Let  $f : A \longrightarrow \mathbb{R}$  and  $g : A \longrightarrow \mathbb{R}$  be two bounded functions such that the closure of  $B = \{x \in A \mid f(x) \neq g(x)\}$  is a set of measure zero. (This implies that  $f$  and  $g$  agree except on a set of measure zero.) Show that  $f$  is integrable if and only if  $g$  is, and that if they are both integrable, then  $\int_A f = \int_A g$ .
6. In class, we considered the set  $A = \mathbb{Q} \cap [0, 1]$  and showed that it has measure zero. In particular, we showed that it was countable, that is, we could write it as  $A = \{a_0, a_1, a_2, a_3 \dots\}$ . Given an  $\varepsilon > 0$ , we then covered it with rectangles  $I_i = [a_i - \frac{\varepsilon}{2^{i+2}}, a_i + \frac{\varepsilon}{2^{i+2}}], \forall i \in \mathbb{N}$ , so that

$$v\left(\bigcup_{i=0}^{\infty} I_i\right) \leq \sum_{i=0}^{\infty} v(I_i) = \varepsilon.$$

Fix  $\varepsilon = \frac{1}{2}$ , and let  $J_i = (a_i - \frac{1}{2^{i+3}}, a_i + \frac{1}{2^{i+3}})$  be the open rectangle equal to the interior of the corresponding  $I_i$  defined above.

Let  $B = \bigcup_{i=2}^{\infty} J_i$ . (Note that we are purposely omitting the first two sets, which cover 0 and 1, respectively, so that each  $J_i \subset [0, 1]$ !)

- (a) Show that  $\partial B = [0, 1] \setminus B$ .
- (b) Show that  $\partial B$  does not have measure zero.
- (c) Let  $\chi_B$  be the characteristic function of  $B$ . Show that  $\chi_B$  is not integrable on  $[0, 1]$ .

(Note that although  $B$  is a “reasonable” set in the sense that it is the union of a countable collection of open sets, it does not have a “reasonable” boundary, and so  $\chi_B$  is not integrable.)