MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 8
Due: *Tuesday*, November 20, 2001

Please turn in five separate sets labelled A through E.

1. Read Sections 1.3, 1.7–1.8 from Edwards.

2. (A) Let $A : V \longrightarrow V$ be a linear transformation on a finite-dimensional vector space, and by slight abuse of notation, let A also be the matrix for this transformation with respect to a fixed basis. Using the following method, we determine the eigenvalues of A:

$$\begin{aligned} \lambda \text{ is an eigenvalue for } A &\iff V_\lambda = Ker(A - \lambda I) \text{ is non-trivial} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff det(A - \lambda I) = 0 \end{aligned}$$

Thus we are inspired to make the following definition:

$p_A(\lambda) = det(A - \lambda I)$ is called the **characteristic polynomial** of A

The eigenvalues of A will be the roots of the characteristic polynomial.

(a) Prove that no scalar $\lambda_0 \in F$ is an eigenvalue for A unless it is a root of $p_A(\lambda)$.

(b) If $p_A(\lambda) = (\lambda - \lambda_0)^k \cdot q(\lambda)$ with $q(\lambda_0) \neq 0$, then we say that the eigenvalue λ_0 has *algebraic multiplicity* equal to k. (That is, λ_0 is a root of $p_A(\lambda)$ of order k.) Show that the geometric multiplicity (which, by definition, is the dimension of the corresponding eigenspace) of an eigenvalue is less than or equal to its algebraic multiplicity.

(c) Use this method to find all eigenvalues of the real matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Is the matrix A diagonalizable? Explain.

3. (A) Show that $A : V \longrightarrow V$ is invertible if and only if $det(A) \neq 0$. (We have used this fact several times already, including in problem #2. The point of this exercise is to make you think carefully about the steps we used when we made the transition from alternating forms to determinants.)

4. (B) Let a and b be real numbers. Use axioms P1–P3 for an ordered field and the definition of absolute value to show that:

   - If $0 < a < b$, then $0 < a^2 < b^2$.
   - If $a^2 < b^2$, then $|a| < |b|$.

   (Note that we have used this fact implicitly, such as when taking square roots, to justify several results about norms and inner products.)

5. (B) Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and an associated norm $|| \cdot ||$. We say that a linear transformation $A : V \longrightarrow V$ is *norm-preserving* if $||A\mathbf{v}|| = ||\mathbf{v}||$ for every $\mathbf{v} \in V$ and *inner-product-preserving* if $\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.

   (a) Show that A is inner-product-preserving if and only if it is norm preserving. (Hint #1: One way is easy! Hint #2: Expand the identity $||A(\mathbf{u} + \mathbf{v})||^2 = ||\mathbf{u} + \mathbf{v}||^2$.)

   (b) Let $V = \mathbb{R}^2$ with the usual inner product. Find all norm-preserving linear transformations/matrices.

6. (Not required) Read Shilov subsections 1.41, 3.13, and 4.53 and Edwards page 36 for the definition of the *transpose* of a matrix. Note that their definitions are equivalent to the following:

   Definition. If $A = [a_{ij}]$ is an $n \times m$ matrix then we define its **transpose** $A^t = [a_{ji}]$ to be the $m \times n$ matrix whose rows are the columns of A. That is,

$$\text{if } A = \begin{bmatrix} a_{11} & \cdot\ \cdot\ \cdot & a_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdot\ \cdot\ \cdot & a_{nm} \end{bmatrix} \text{ then } A^t = \begin{bmatrix} a_{11} & \cdot\ \cdot\ \cdot & a_{n1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{1m} & \cdot\ \cdot\ \cdot & a_{mn} \end{bmatrix}.$$

   Note that this definition applies equally well to vectors, which may be considered as $n \times 1$ matrices. Finally, read the three most important theorems concerning transposed matrices, and convince yourself of their validity.

   Theorem: If A and B are $n \times m$, then $(A + B)^t = A^t + B^t$.

   Theorem: If A is $n \times m$ and B is $m \times k$, then $(AB)^t = B^t A^t$.

   Theorem: If A is $n \times n$, then $det(A^t) = det(A)$.

7. (C) Let $V = \mathbb{R}^n$, and let $\mathbf{u}, \mathbf{v} \in V$. If $A : V \longrightarrow V$ is a linear transformation, then define the following bilinear form:

$$f_A(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v}$$

   (a) (Not required) Convince yourself that f_A is indeed a bilinear form.

   (b) Give a necessary and sufficient condition on A that makes f_A an inner product. (Full points for a complete answer in the $n = 2$ case.)

   (c) Give a necessary and sufficient condition on A that makes f_A alternating.

   (Hint #1: You might consider the $n = 2$ case first to get a feel for this bilinear form. Hint #2: Your answers just might involve the transpose!)

8. (D) (Read Example 8 and problem 3.12 in Section 1.3 of Edwards.)

   (a) Consider the vector space $V = C[-1, 1]$ of continuous functions on the closed interval $[-1, 1]$ over the real numbers with inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

   Orthogonalize the set of functions $\{1, x, x^2, x^3, x^4\}$ with respect to this inner product.

   (b) With $V = C[0, 1]$ and inner product

$$\langle f(x), g(x) \rangle = \int_{0}^{1} f(x)g(x)\, dx,$$

   orthogonalize the same set of functions from part (a).

   (c) Now let $V = C[-1, 1]$ as in part (a), but define $\langle \cdot, \cdot \rangle$ as in part (b). Show that in fact this bilinear form is *not* an inner product on V.

9. (E) (Read Example 3 in Section 1.3 of Edwards.)

   Show that the function $|| \cdot || : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $||(x_1, \ldots, x_n)|| = \max\{|x_1|, \ldots, |x_n|\}$ defines a norm on $\mathbb{R}^n$. Show that there is no inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that this norm is associated to the inner product.

10. (E) Recall that in HW #1.10, we put an unusual absolute value on the rational numbers $\mathbb{Q}$. We now generalize this procedure to any fixed prime number $p \in \mathbb{N}$. For any non-zero $x \in \mathbb{Q}$, write $x = p^n \cdot \frac{a}{b}$, where $\frac{a}{b}$ is in lowest terms (that is, a and b have a greatest common divisor of 1) and p divides neither a nor b. Define a new absolute value $|\cdot|_p$ on $\mathbb{Q}$ as follows:

$$|x|_p = p^{-n}$$

We also define $|0|_p = 0$.

This absolute value satisfies the following properties:

i. $|x|_p > 0$, for all non-zero $x \in \mathbb{Q}$

ii. $|xy|_p = |x|_p \cdot |y|_p$, for all $x, y \in \mathbb{Q}$

iii. $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, for all $x, y \in \mathbb{Q}$

(The proof that you used for $p = 2$ works in this case as well. Recall that property iii. is called the Ultra-metric Inequality and is stronger than the Triangle Inequality.)

Having defined this new absolute value, we define the new set of numbers $\mathbb{Q}_p$, the p-adics, to be the set of equivalence classes of Cauchy sequences of rational numbers with respect to this absolute value. We extend the definition of $|\cdot|_p$ to all of $\mathbb{Q}_p$ by continuity. That is, if $\{a_n\}$ is a Cauchy sequence representing the equivalence class of the p-adic number x, then we define $|x|_p = \lim_{n\to\infty} |a_n|_p$. This limit always converges because $\{a_n\}$ is Cauchy, and it may be verified that this absolute value is also well-defined. Finally, with the usual definitions of addition and multiplication (which are also well-defined), one checks that $\mathbb{Q}_p$ is a complete field with respect to this absolute value.

Finally, for this problem, we let $V = (\mathbb{Q}_p)^2$ be the vector space of ordered pairs of p-adic numbers. Show that the map $||\cdot||_p : V \longrightarrow \mathbb{Q}$ given by

$$||(x, y)||_p = \max\{|x|_p, |y|_p\}$$

defines a norm on V.