MATH 23b, SPRING 2002
THEORETICAL LINEAR ALGEBRA AND MULTIVARIABLE CALCULUS
Homework Assignment # 8

Due: April 19, 2002

Homework Assignment #8 (Final Version)

1. Read Edwards Sections 4.3–4.5, including problems 4.3.5–4.3.8.

2. Let $A \subset \mathbb{R}^n$ be bounded, and let $\mathcal{I} = \{f : A \longrightarrow \mathbb{R} | f \text{ is bounded}\}$. It is easy to see that $\mathcal{I}$ is a vector space. We define a function $||\cdot||_\infty : \mathcal{I} \longrightarrow \mathbb{R}$ by the following:
$$||f||_\infty = \sup_{x \in A}\{|f(x)|\}.$$
Show that $||\cdot||_\infty$ is a norm on $\mathcal{I}$.

3. Consider the subspace $I \subset \mathcal{I}$ defined as $I = \{f : A \longrightarrow \mathbb{R} | f \text{ is integrable on } A\}$. We define a function $||\cdot||_1 : I \longrightarrow \mathbb{R}$ by:
$$||f||_1 = \int_A |f|.$$
The function $||\cdot||_1$ satisfies the rules for being a norm on I except that there exist functions $f \neq 0$ such that $||f||_1 = 0$; in other words, this function is positive but not positive-definite. There are two possible modifications we can make:

   (a) Let $J = \{f \in I | f \text{ is continuous on } A\}$ be a subspace of I. Show that $||\cdot||_1$ is positive definite on J by showing that if $f \neq 0$, then $||f||_1 > 0$.

   (b) Let $K = \{f \in I | \int_A |f| = 0\}$ be a subspace of I. If we define the quotient space I/K, then two functions $f, g \in I$ are equivalent if $f - g \in K$, that is, if $\int_A |f - g| = 0$. Show that $||\cdot||_1$ defined on I/K by $||f + K||_1 = \int_A |f|$ is well-defined and positive definite.

4. For each $n \in \mathbb{N}$, consider the function $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:
$$f_n(x) = (1/n)e^{-n^2x^2}.$$

   (a) Show that $f_n \to 0$ uniformly on $\mathbb{R}$.

   (b) Show that $f_n' \to 0$ pointwise on $\mathbb{R}$.

   (c) Show that f_n' does not converge uniformly to 0 on any interval containing the origin.

5. Let $A \subset \mathbb{R}^n$ be bounded, and for each $n \in \mathbb{N}$, let $f_n : A \longrightarrow \mathbb{R}$ be integrable. Suppose $f_n \to f$ uniformly on A. Show that $\left\{ \int_A f_n \right\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers.

6. Problem # 4.5 from p. 243 of Edwards:

   Let $S \subset \mathbb{R}^3$ be the (bounded) intersection of the two (unbounded) cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. Show that the volume of S is $\frac{16}{3}$.

7. Problem # 4.7 from p. 244 of Edwards:

   Show that the volume of $B_1(0) \subset \mathbb{R}^4$ is $\frac{\pi^2}{2}$.

   (Hint: Use the fact that the volume of a ball in $\mathbb{R}^3$ of radius r is $\frac{4}{3}\pi r^3$.)