MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 1

Construction of the Integers, $\mathbb{Z}$

Given the natural numbers, $\mathbb{N}$, and their arithmetic operations $+$ and $\cdot$, we construct the integers as follows:

Let $\mathbb{Z} = \{(a, b)|a, b \in \mathbb{N}\}/\sim$, where $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

Our rationale for doing so is to introduce additive inverses for the natural numbers by defining -1 to be the equivalence class containing all pairs whose "differences" are -1, such as $(0, 1)$, $(1, 2)$, etc. Note that according to this definition, 0 corresponds to the equivalence class of all pairs of the form (n, n), and in general, the integer m corresponds to the equivalence class of all pairs of the form $(n + m, n)$.

Thus the elements of $\mathbb{Z}$ are equivalence classes, and we will use the notation $[(a, b)]$ to denote the equivalence class of the pair (a, b). Given our set $\mathbb{Z}$, we must define the operations of addition and multiplication. We first consider addition. Thinking of the pairs by the differences they represent, adding (a, b) to (c, d) is the same as adding $a - b$ to $c - d$, and so we want the result to be $(a - b) + (c - d) = (a + c) - (b + d)$. Thus, it makes sense to try defining:

Addition of two integers: $[(a, b)] + [(c, d)] = [(a + c, b + d)]$

However, we should verify that this definition does not depend on the choice of representative from the equivalence class. That is, suppose $(a', b') \in [(a, b)]$ and $(c', d') \in [(c, d)]$. Then we must show that the sum of these representatives is in the equivalence class $[(a + c, b + d)]$. We do this as follows:

> If $(a', b') \in [(a, b)]$, then $a' + b = a + b'$, and similarly, if $(c', d') \in [(c, d)]$, then $c' + d = c + d'$. Hence, by adding the two left-hand members and the two right-hand members and re-arranging by associativity, we obtain:
>
> $$(a' + c') + (b + d) = (a + c) + (b' + d')$$
>
> According to the equivalence relation, another way to read this equality is to say that $(a' + c', b' + d') \sim (a + c, b + d)$, and thus the result of adding (a', b') to (c', d') is indeed in the correct equivalence class, and we say that the addition of two integers is now *well-defined*.