MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 2, supplement

Construction of the Rational Numbers, $\mathbb{Q}$

Given the integers, $\mathbb{Z}$, and their arithmetic operations $+$ and $\cdot$, we construct the rational numbers as follows:

Let $\mathbb{Q} = \{(a,b) | a,b \in \mathbb{Z}, \text{ and } b \neq 0\}/\sim$, where $(a,b) \sim (c,d)$ if and only if $a \cdot d = b \cdot c$.

Our rationale for doing so is to introduce multiplicative inverses for the integers by defining $\frac{1}{2}$ to be the equivalence class containing all pairs whose "quotients" are $\frac{1}{2}$, such as $(1,2)$, $(2,4)$, $(-3,-6)$, etc. Note that according to this definition, 0 corresponds to the equivalence class of all pairs of the form $(0,n)$, 1 corresponds to the equivalence class of all pairs of the form (n,n), and in general, the rational number $\frac{p}{q}$ corresponds to the equivalence class of all pairs of the form $(p \cdot n, q \cdot n)$.

Thus the elements of $\mathbb{Q}$ are equivalence classes, and we will use the notation $[(a,b)]$ to denote the equivalence class of the pair (a,b).

Given our set $\mathbb{Q}$, we must define the operations of addition and multiplication. Addition we define by the rule:

$$[(a,b)] + [(c,d)] = [(ad+bc, bd)]$$

and multiplication we define by the rule:

$$[(a,b)] \cdot [(c,d)] = [(ac, bd)]$$

We leave as exercises the verification that these operations are well-defined. We must also prove as theorems the usual rules of commutativity and associativity for both operations, the distributive law for the interaction of the two operations, and the existence of additive and multiplicative identities and inverses. (Of course, the additive identity will have no multiplicative inverse.) Note that we have named the two identities, 0 and 1, above, but it should be verified that they indeed *act* as identities.