MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 3, supplement

The Axioms for an Ordered Field

An *ordered field* F is a set with two closed binary operations $+$ (addition) and $\cdot$ (multiplication) and a subset P (the positive elements) satisfying the following:

A1. Commutativity of Addition: $a + b = b + a,\ \forall a, b \in F$

A2. Associativity of Addition: $(a + b) + c = a + (b + c),\ \forall a, b, c \in F$

A3. Existence of an Additive Identity: There exists a unique element $0 \in F$ such that $0 + a = a,\ \forall a \in F$.

A4. Existence of Additive Inverses: For every $a \in F$, there exists an element $b \in F$ such that $a + b = 0$. We usually denote this element by $-a$.

M1. Commutativity of Multiplication: $a \cdot b = b \cdot a,\ \forall a, b \in F$

M2. Associativity of Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c),\ \forall a, b, c \in F$

M3. Existence of a Multiplicative Identity: There exists a unique element $1 \in F$ such that $1 \cdot a = a,\ \forall a \in F$.

M4. Existence of Multiplicative Inverses: For every $a \in F$ except $a = 0$, there exists an element $b \in F$ such that $a \cdot b = 1$. We usually denote this element by a^{-1}.

D. Distributivity: $a \cdot (b + c) = (a \cdot b) + (a \cdot c),\ \forall a, b, c \in F$

P1. Closure of the Positive Elements under Addition:

If $a, b \in P$, then $a + b \in P$.

P2. Closure of the Positive Elements under Multiplication:

If $a, b \in P$, then $a \cdot b \in P$.

P3. Trichotomy: If $a \in F$, then precisely one of the following is true:

- $a \in P$
- $-a \in P$
- $a = 0$

Facts about Ordered Fields

In all of the following, we consider an ordered field F with additive identity 0, multiplicative identity 1, and positive elements P.

- Fact: $a \cdot 0 = 0, \ \forall a \in F$
- Fact: Additive inverses are unique. That is, if $a \in F$, then there is one and only one element $b \in F$ such that $a + b = 0$.
- Fact: Multiplicative inverses are unique. That is, if $a \in F$ and $a \neq 0$, then there is one and only one element $b \in F$ such that $a \cdot b = 1$.
- Fact: $-(a+b) = (-a) + (-b), \ \forall a, b \in F$
- Fact: $(-a) \cdot (-b) = a \cdot b, \ \forall a, b \in F$

Definition:

$$a < b \text{ iff } b - a \in P$$

- Fact: If $a < b$, then $a + c < b + c, \ \forall c \in F$
- Fact: If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$.
- Fact: If $a < b$ and $c < 0$, then $a \cdot c > b \cdot c$.

Definition:

$$|a| = \begin{cases} a & , \text{ if } a \in P \\ 0 & , \text{ if } a = 0 \\ -a & , \text{ if } -a \in P \end{cases}$$

- Fact: $|a \cdot b| = |a| \cdot |b|, \ \forall a, b \in F$
- Fact: $|a + b| \leq |a| + |b|, \ \forall a, b \in F$