

MATH 23a, FALL 2001  
THEORETICAL LINEAR ALGEBRA  
AND MULTIVARIABLE CALCULUS  
Lecture # 22, supplement

Alternating and Skew-Symmetric Forms

**Definition:** A multilinear form  $f : V^k \longrightarrow F$  is said to be *alternating* if and only if:

$$f(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_k) = 0$$

whenever  $\mathbf{v}_i = \mathbf{v}_j$  with  $i \neq j$ .

**Definition:** A multilinear form  $f : V^k \longrightarrow F$  is said to be *skew-symmetric* if and only if:

$$f(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_k) = -f(\mathbf{v}_1, \dots, \mathbf{v}_j, \dots, \mathbf{v}_i, \dots, \mathbf{v}_k)$$

for every pair  $i \neq j$ .

**Theorem:** If the multilinear form  $f : V^k \longrightarrow F$  is alternating, then it is skew-symmetric.

**Proof:** For simplicity of notation but without any loss of generality, we give the proof for  $k = 2$ . Note that since  $f$  is alternating, we use the fact that  $f(\mathbf{v}, \mathbf{v}) = 0$  for any  $\mathbf{v} \in V$  in several of the steps below:

$$\begin{aligned} f(\mathbf{v}_1, \mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_1) &= [f(\mathbf{v}_1, \mathbf{v}_2) + f(\mathbf{v}_1, \mathbf{v}_1)] + [f(\mathbf{v}_2, \mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_1)] \\ &= f(\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2) + f(\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) \\ &= f(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) \\ &= 0 \end{aligned}$$

This implies that  $f(\mathbf{v}_1, \mathbf{v}_2) = -f(\mathbf{v}_2, \mathbf{v}_1)$ .