MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 32, supplement

Connectedness

Definition: A set $S \subset \mathbb{R}^n$ is said to be **disconnected** if there exist disjoint open sets A and B such that $S \cap A \neq \emptyset$, $S \cap B \neq \emptyset$ and $S \subset A \cup B$. (In other words, S is covered by A and B in such a way that A and B each actually do some of the covering, and A and B do not interesect.) S is called **connected** if it is not disconnected.

Definition: A set $S \subset \mathbb{R}^n$ is said to be **path-connected** if, given any $a, b \in S$, there exists a continuous function $f : [0, 1] \longrightarrow S$ such that $f(0) = a$ and $f(1) = b$.

Example: $\mathbb{Q}$ is disconnected because we can choose $A = (-\infty, \pi)$ and $B = (\pi, +\infty)$, which are open and disjoint and which satisfy the required conditions.

Example: The Topologist's Sine Curve is the set

$$S = \left\{ (x, 0) \,\middle|\, x \leq 0 \right\} \cup \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \,\middle|\, x > 0 \right\}.$$

It is connected but not path-connected.

Lemma: All intervals in $\mathbb{R}$ are connected.

Proof: This is just long enough to merit its own discussion separately.

Theorem: If $S \subset \mathbb{R}^n$ is path-connected, then it is connected.

Proof: Suppose S is disconnected. Let A and B be the disjoint open sets given in the definition, and choose $a \in A$ and $b \in B$.

Suppose $f : \mathbb{R} \longrightarrow S$ is a continuous function such that $f(0) = a$ and $f(1) = b$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are open in $\mathbb{R}$ (by the continuity of f), and since $0 \in f^{-1}(A)$ and $1 \in f^{-1}(B)$, their intersections with $[0, 1]$ are non-empty. They are disjoint by definition, and since $S \subset A \cup B$, it is clear that $[0, 1] \subset \mathbb{R} \subset f^{-1}(A) \cup f^{-1}(B)$. But these conditions would imply that $[0, 1]$ is disconnected, which contradicts the lemma, and hence any such f must not be continuous.

Hence S is not path-connected, and we have proven the contrapositive.

Theorem: If $S \subset \mathbb{R}^n$ is connected and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous, then $f(S)$ is connected.

> **Proof:** Suppose $f(S)$ is disconnected. Let A and B be disjoint open sets such that $f(S) \cap A \neq \emptyset$, $f(S) \cap B \neq \emptyset$, and $f(S) \subset A \cup B$, as in the definition.
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> Define $g : f(S) \longrightarrow \mathbb{R}$ by $g(x) = 0, \forall x \in A \cap f(S)$ and $g(x) = 1, \forall x \in B \cap f(S)$. Then $g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous.
>
> Choose $0 < \varepsilon < \frac{1}{2}$. Let $A' = (g \circ f)^{-1}(0) = (g \circ f)^{-1}(-\varepsilon, +\varepsilon)$ and let $B' = (g \circ f)^{-1}(1) = (g \circ f)^{-1}(1-\varepsilon, 1+\varepsilon)$. Then A' and B' are open since the intervals are open and since $f \circ g$ is continuous, and they are disjoint by definition since the intervals do not intersect. It is also clear that $S \subset A' \cup B'$. Finally, $S \cap A'$ is non-empty since $f(S) \cap A$ is non-empty, and a similar statement may be made for $S \cap B'$.
>
> This shows that S is disconnected, thus proving the contrapositive.

Theorem (The Intermediate Value Theorem): Let $S \subset \mathbb{R}^n$ be connected, and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous. Let $x, y \in S$, and suppose $f(x) \leq c \leq f(y)$. Then there exists $z \in S$ such that $f(z) = c$.

> **Proof:** Suppose not. Then $f(S)$ is disconnected because we can choose $A = (-\infty, c)$ and $B = (c, +\infty)$, two disjoint open sets satisfying the definition.