MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Quiz # 2 (Solutions)
December 10, 2001

False or False

All of the following statements are *false*. Give alternate versions of the statements that are true, and give examples of why the original statements are false.

1. The intersection of every collection of open sets is open.

   - There are three closely related true statements:
     i. The intersection of every collection of *closed* sets is *closed*.
     ii. The *union* of every collection of open sets is open.
     iii. The intersection of *finitely many* open sets is open.
   - From the third true statement above, we know that any counter-example must have infinitely many sets. A few such examples:
     i. In $\mathbb{R}^n$, we have $\bigcap_{n=1}^{\infty} B_{1/n}(\mathbf{0}) = \{\mathbf{0}\}$, where the balls are open and the point is closed.
     ii. In $\mathbb{R}$, we have $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, 1 + \frac{1}{n}) = [0, 1]$.

2. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $A \subset \mathbb{R}^n$ is closed, then $f(A)$ is closed in $\mathbb{R}^m$.

   - Two closely related true statements:
     i. If $A \subset \mathbb{R}^m$ is closed, then $f^{-1}(A)$ is closed in $\mathbb{R}^n$.
     ii. If $A \subset \mathbb{R}^n$ is compact, then $f(A)$ is compact in $\mathbb{R}^m$.
   - For any counter-example, we must produce a function f and a set A. Since we know the second true statement above, any such example must have A unbounded, since otherwise, A would be compact, and so $f(A)$ would also be compact, and hence $f(A)$ would be closed.

     Four examples, then:
     i. $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x}$ with $A = [1, +\infty)$. Then $f(A) = (0, 1]$, which is not closed. It's not open either, but that is not required!
     ii. Same example with $A = \mathbb{N}$. Then $f(A) = \{\frac{1}{n} | n \in \mathbb{N}\}$, which has 0 as a limit point. Since 0 is not in the set, $f(A)$ is not closed.

iii. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $f(x) = e^x$. If $A = \mathbb{R}$, then $f(A) = (0, +\infty)$ is open.

iv. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by

$$f(\mathbf{x}) = \frac{1}{||\mathbf{x}||^2 + 1}.$$

If $A = \mathbb{R}^n$, then $f(A) = (0, 1]$.

3. If $B \subset \mathbb{R}^n$ is closed, then every open cover of B has a finite subcover.

- The only closely related true statement is that if $B \subset \mathbb{R}^n$ is compact, then every open cover of B has a finite subcover.
- If B is bounded as well as closed, then B will be compact, so any counter-example must use an unbounded B. Also, any correct counter-example must actually exhibit a cover with no finite subcover. Many people gave a set B and then claimed that it had no finite subcover, but this begs the question: subcover of what original cover? Note that *every* set does have a finite cover, namely by the whole space, considered as a single set.

  Here are two counter-examples to the original:

  i. Let $B = \mathbb{R}^n$ itself, and take the cover $B \subset \bigcup_{n=1}^{\infty} B_n(\mathbf{0})$.

  ii. Let $B = \mathbb{N} \subset \mathbb{R}$, and consider the cover $B \subset \bigcup_{n=1}^{\infty} (n - \frac{1}{4}, n + \frac{1}{4})$.