MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam (take-home portion)
Due: January 12, 2003

Directions: For this exam, you will work in teams of four students, which should be confirmed with the instructor before beginning the exam. You have until 5 P. M. on Sunday, January 12, to complete this exam, when it must be turned in under my door, SC 320. You may use your own class notes (or those of your teammates), your own homework assignments (or those of your teammates), and the course textbooks (and supplemental bibliography) as your only aids. You may not use any internet resources except for the course website and the posted homework and exam solutions.

You may discuss the exam only with your teammates, and all questions should be directed only to the instructor. Obviously, you will be studying for the in-class portion of the exam concurrently, and so you will be able to speak with the Math 23a CA's, but you should not put them in an awkward position by asking questions directly related to the take-home, and they will be instructed to report anyone attempting to gain unfair advantage.

Each team should submit one solution set, with each team member writing a solution to one of the problems. There is partial credit, but only for intelligible work. One point per problem will be awarded for *neatness only*, and one point will be awarded for *style only*. Make sure your name and your teammates' names are prominently displayed on your work, and *please* staple your final pages together into one stack.

You may quote results from class and/or your notes with an appropriate reference, and you must cite anything you take from a textbook. Otherwise, all work should be that of you and your team.

1. The group $GL_n(\mathbb{R})$

Consider the vector space $V = \mathbb{R}^n$. We define the collection of linear transformations (and their matrices with respect to the standard basis) from V to V to be:

$$M_n(\mathbb{R}) = \{A : V \longrightarrow V \mid A \text{ is linear}\}.$$

We further define the collection (which has the algebraic structure of a *group*) of invertible linear transformations, called the *general linear group*, to be:

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A \text{ is invertible}\}.$$

(a) Show that the determinant function,

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$

is continuous.

(b) Show that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$. (Hint: Use part (a).)

(c) Show that the closure of $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is all of $M_n(\mathbb{R})$.

2. Nilpotent Matrices

Let $V = \mathbb{R}^n$, and let $A : V \to V$. We say that A is *nilpotent* if there is some $m \in \mathbb{N}$ such that $A^m = 0$, and we say that m is the *degree* of nilpotency if $A^m = 0$ but $A^{(m-1)} \neq 0$.

(a) Let $A : V \to V$ be nilpotent of degree m. Let $\mathbf{v} \in V$ be such that $A^{(m-1)}\mathbf{v} \neq \mathbf{0}$. Show that $\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots, A^{(m-1)}\mathbf{v}\}$ is a linearly independent set in V. Conclude that $m \leq n$.

(b) Exhibit examples of nilpotent matrices of degrees 0,1,2, and 3.

(c) Let $A : V \to V$ be nilpotent. Find all the eigenvalues of A.

(d) Show that if A is nilpotent, then $I + A$ is invertible.
(If A is nilpotent, then $I + A$ is called *unipotent*.)

3. "Space-Time"

Let $V = \mathbb{R}^3$, and consider the bilinear map $f : V \times V \to \mathbb{R}$ given by

$$f(\mathbf{u},\mathbf{v}) = \langle \mathbf{u},\mathbf{v}\rangle = \mathbf{u}^t B\mathbf{v}, \quad \text{where} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This form f defines a new geometry on V, which has orthogonality defined by $\langle \mathbf{u},\mathbf{v}\rangle = 0$ and length defined by $||\mathbf{u}|| = \sqrt{|\langle \mathbf{u},\mathbf{u}\rangle|}$.

This geometry differs from Euclidean geometry because we find that there are points/vectors whose behaviors vary according to f:

- If $\langle \mathbf{u},\mathbf{u}\rangle > 0$, we say that $\mathbf{u}$ is space-like.
- If $\langle \mathbf{u},\mathbf{u}\rangle = 0$, we say that $\mathbf{u}$ is light-like.
- If $\langle \mathbf{u},\mathbf{u}\rangle < 0$, we say that $\mathbf{u}$ is time-like.

Because we still have length and orthogonality, we may still speak of an "orthonormal basis" and the "orthogonal complement" of a subspace $W \subset V$, but now defined in terms of f.

(a) Show that f is not an inner product.

(b) Show that f is not an alternating form.

(c) Show that any orthonormal basis for V must consist of two space-like vectors and one time-like vector.

(d) Let $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, and let $W = \text{span}\{\mathbf{w}\}$. Find a basis for $W^\perp$ and show that $W \cap W^\perp \neq \{\mathbf{0}\}$.

4. Upper Triangular Matrices and an Application to Differential Equations

Consider $V = \mathbb{R}^2$, and as in problem #1, define

$$M_2(\mathbb{R}) = \{A : V \longrightarrow V | \ A \text{ is linear}\}.$$

We have seen that if $A \in M_2(\mathbb{R})$ has two distinct real eigenvalues, say λ_1 and λ_2, then A is diagonalizable, and there is some basis for V with respect to which A may be diagonalized and written in the form $S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, for some invertible matrix S. We have also seen examples of matrices in $M_2(\mathbb{R})$, such as rotations like $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, that have no real eigenvalues.

(a) Show that if $A \in M_2(\mathbb{R})$ has exactly one real eigenvalue, say λ_0, then there is some basis for V with respect to which A may be diagonalized and written in the form $S^{-1}AS = \begin{bmatrix} \lambda_0 & b \\ 0 & \lambda_0 \end{bmatrix}$.

(b) (Not required) Observe that any such matrix A (from part (a)) may be decomposed as follows:

$$\underbrace{\begin{bmatrix} \lambda_0 & b \\ 0 & \lambda_0 \end{bmatrix}}_{S^{-1}AS} = \underbrace{\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}}_{N},$$

where D is a diagonal matrix, and N is a nilpotent matrix (see problem #2). Furthermore, note that $DN = ND$.

Now we consider an application to linear differential equations. Consider the second-order differential equation:

$$y'' + ay' + by = 0, \quad \text{where } a, b \in \mathbb{R}.$$

A solution to this equation is a function y, considered to have the single variable t. This equation is called an *ordinary* differential equation because there is only one independent variable for the function y. It is *second-order* because it involves the second-derivative of y and no higher order derivatives. It is said to be *homogeneous* becuase every term involves y or one of its derivatives.

We use linear algebra to solve this equation in the following manner. (See Curtis, Section 34, for more information, which we only summarize here!) Let $v_1 = y$ and let $v_2 = y'$. Then we can express this differential equation as a system of linear differential equations:

$$\begin{aligned} \frac{dv_1}{dt} &= v_2 \\ \frac{dv_2}{dt} &= -av_2 - bv_1 \end{aligned}$$

Note that we can write this system as a matrix equation:

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

or in other words, $\dfrac{d\mathbf{v}}{dt} = A\mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Suprisingly, this equation has the solution $\mathbf{v}(t) = \mathbf{v}_0 \cdot e^{At}$ (where $\mathbf{v}_0$ is some constant vector depending on initial conditions) provided that we can make sense of the exponential of a matrix.

(Note the analogy with the first-order differential equation $\frac{dy}{dx} = ay$, which has general solution $y(x) = C \cdot e^{ax}$, where C is some real constant.)

We use the Taylor series expansion for the function e^x about $x = 0$, which is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

In the case of the matrix A, we *define*:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots,$$

provided this series converges, which in fact it always does. (For a more precise definition, see Curtis, Definition 34.5.)

A practical computation of e^A can be difficult, however. Fortunately, when A has at least one real eigenvalue, we can put parts (a) and (b) to work for us and write $S^{-1}AS = D + N$. Then we use some facts (only one of which you need to verify):

- $e^{A+B} = e^A \cdot e^B$, provided that $AB = BA$
- $S^{-1}e^A S = e^{(S^{-1}AS)}$, for any invertible S
- If $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $e^D = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$.
- If N is nilpotent, then there is some $m \in \mathbb{N}$ such that $N^m = 0$, and hence the expression $e^N = 1 + N + \frac{N^2}{2!} + \cdots + \frac{N^{(m-1)}}{(m-1)!}$ is a finite sum!

Putting all of this information together enables us to find the following particular solution to the original system:

$$\mathbf{v}(t) = Se^{tD}e^{tN}S^{-1}\mathbf{v}_0,$$

where $\mathbf{v}_0 = \mathbf{v}(0)$ is a set of initial conditions.

(c) Verify the second fact above, i. e. $S^{-1}e^A S = e^{(S^{-1}AS)}$, for any invertible S.

(d) Use these techniques to find the particular solution to the differential equation $y'' - y' - 6y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 1$.

(e) Use these techniques to find the particular solution to the differential equation $y'' - 4y' + 4y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 1$.