MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment #10
Due: December 13, 2002

1. Read Sections 18-19 of Chapter 5 in Curtis and Sections 10.2 and 10.3 of Fitzpatrick.

2. (A) Recall the definition of the *transpose* of a matrix, as referred to in homework problem #8.8, and prove the following:

   **Theorem:** If $A$ is an $n \times n$ matrix, then $det(A^t) = det(A)$.

3. (B) Let $A : V \longrightarrow V$ be a linear transformation on a finite-dimensional vector space, and by slight abuse of notation, let $A$ also be the matrix for this transformation with respect to a fixed basis. Using the following method, we determine the eigenvalues of $A$:

$$\begin{aligned} \lambda \text{ is an eigenvalue for } A \iff & V_\lambda = Ker(A - \lambda I) \text{ is non-trivial} \\ \iff & A - \lambda I \text{ is not invertible} \\ \iff & det(A - \lambda I) = 0 \end{aligned}$$

   Thus we are inspired to make the following definition:

   $p_A(\lambda) = det(A - \lambda I)$ is called the **characteristic polynomial** of $A$

   The eigenvalues of $A$ will be the roots of the characteristic polynomial.

   (a) Prove that no scalar $\lambda_0 \in F$ is an eigenvalue for $A$ unless it is a root of $p_A(\lambda)$.

   (b) If $p_A(\lambda) = (\lambda - \lambda_0)^k \cdot q(\lambda)$ with $q(\lambda_0) \neq 0$, then we say that the eigenvalue $\lambda_0$ has *algebraic multiplicity* equal to $k$. (That is, $\lambda_0$ is a root of $p_A(\lambda)$ of order $k$.) Show that the geometric multiplicity (which, by definition, is the dimension of the corresponding eigenspace) of an eigenvalue is less than or equal to its algebraic multiplicity.

   (c) Use this method to find all eigenvalues of the real matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

   Is the matrix $A$ diagonalizable? Explain.

4. (A) Show that $A : V \longrightarrow V$ is invertible if and only if $det(A) \neq 0$. (We have used this fact several times already, including in problem #3. The point of this exercise is to make you think carefully about the steps we used when we made the transition from alternating forms to determinants.)

5. (C) It is a fact that if $A$ and $B$ are $n \times n$ matrices over the same field $F$, then $\det(AB) = \det(A) \cdot \det(B)$. Curtis proves this fact in Section 18 and sketches another proof in problems 3 and 4 of that section.

   (a) Read Curtis, Section 18, especially Lemma 18.1 and Theorem 18.3.

   (b) Read Example C from Curtis, Section 12, to understand *elementary matrices*.

   (c) Show that every invertible $n \times n$ matrix A can be written as a product of elementary matrices. *(Hint: Think about the row reduction process.)*

   (d) Compute the determinant of each of the three types of elementary matrices.

   (e) Prove the fact stated above in three steps, more or less as outlined in Curtis' problem #4:

      i. Show that if either $A$ or $B$ is not invertible, then $\det(AB) = \det(A) \cdot \det(B)$ because both sides are zero.

      ii. Show that if $E$ is an elementary matrix and $B$ is any matrix, then $\det(EB) = \det(E) \cdot \det(B)$ by considering the relationship bewteen $B$ and $EB$ and using part (d) above.

      iii. Use part (c) above to finish the proof.

6. (B) Let $V$ be a real vector space with a convergent sequence $\{\mathbf{v}_n\}$ such that $\lim_{n\to\infty} \mathbf{v}_n = \mathbf{v}$, and let $\{c_n\}$ be a convergent sequence of real numbers with $\lim_{n\to\infty} c_n = c$. Show that $\{c_n\mathbf{v}_n\}$ converges to $c\mathbf{v}$.

7. (D) Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous at $\mathbf{a}$, and suppose $g(\mathbf{a}) \neq 0$. Show that $\dfrac{f}{g}$ is continuous at $\mathbf{a}$.

8. (D) Define $f : [0,1] \longrightarrow \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0 & , \text{ if } x \notin \mathbb{Q} \\ \frac{1}{q} & , \text{ if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

   (a) Graph $f$.

   (b) Show that $f$ is not continuous at any rational $x$.

   (c) Show that $f$ is continuous at any irrational $x$.

9. (deferred) Given a set $S$ in a normed vector space, we define a point $x \in S$ to be an **interior point** if $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset S$. We define the **interior** of $S$ to be the set of interior points, and we denote it by $S^\circ$. Show that the interior of any set is open.

10. (deferred) Let $V = \mathbb{R}^2$, and consider the following subsets:

    $A = \mathbb{Q} \times \mathbb{Q} = \{(x, y) \in \mathbb{R}^2 | x, y \in \mathbb{Q}\}$ and $B = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$

    For the following, recall that $\overline{S}$, $S^\circ$, and $S^c$ denote the closure, interior (see #8), and complement, respectively, of $S$.

    (a) Find $A^\circ$, $(A^c)^\circ$, $\overline{A}$, $\overline{A^c}$, and $\overline{A} \cap \overline{A^c}$.

    (b) Find $B^\circ$, $(B^c)^\circ$, $\overline{B}$, $\overline{B^c}$, and $\overline{B} \cap \overline{B^c}$.

    (c) Find $\overline{A \cap B}$ and $(A \cap B)^\circ$.

11. (deferred) Let $S = \{(x, \sin(\frac{1}{x})) \mid x > 0\} \subset \mathbb{R}^2$. Find $\overline{S}$.

12. (deferred) Let $\{S_n\}$ be a collection of open sets in a normed vector space, and let $\{T_n\}$ be a collection of closed sets. Show that:

    (a) $S_1 \cap S_2$ is open.

    (b) $\bigcup S_n$ is open.

    (c) $T_1 \cup T_2$ is closed.

    (d) $\bigcap T_n$ is closed.