Math 23b, 2003.
Solution Set 10, Questions 3,4.
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Question 3. Let $F(x, y, z) = (z^3 + 2xy, x^2 + 1, 3xz^2)$ be a vector field on $\mathbb{R}^3$. Show that F is conservative by computing the partial derivatives of its component functions, and find $f : \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$.

Answer. Let's do that cross-partials thing the theorem tells us to do. Call the components of F with subscripts 1, 2 and 3.

$$\frac{\partial F_1}{\partial y} = 2x \qquad \frac{\partial F_1}{\partial z} = 3z^2 \qquad \frac{\partial F_2}{\partial z} = 0$$
$$\frac{\partial F_2}{\partial x} = 2x \qquad \frac{\partial F_3}{\partial x} = 3z^2 \qquad \frac{\partial F_3}{\partial y} = 0.$$

And they match! Hurrah. So f is continuously differentiable with $\nabla f = F$. Now to recover f, we take it component at a time. Since $F_1 = \partial f/\partial x$, we know that the original function looked like $f = xz^3 + x^2y + C(y, z)$ where $C(y, z)$ is our "constant of integration." But when we skip to the second and third components, we learn that $\partial C(y, z)/\partial y = 1$ and 0 with respect to z. Conclusion: $C(y, z) = y + k$, where k is a bona fide constant. All in all, $F(x, y, z) = \nabla f(x, y, z) = \text{grad}(xz^3 + x^2y + y + k)$.

Question 4. Evaluate $\int_C 2xyz\, dx + x^2z\, dy + x^2y\, dz$, where C is a piece-wise smooth oriented curve from $(1, 1, 1)$ to $(1, 2, 4)$.

Answer. Since we weren't given an explicit path C, we better hope that vector field $(2xyz, x^2z, x^2y)$ is conservative. And it is! Let's go looking for *any one particular* potential. By inspection, $\nabla f(x, y, z) = \text{grad}(x^2yz) = F(x, y, z)$ screams for attention. Then by the Fundemental Theorem of Line Integrals, if $\gamma : [a, b] \to C = \text{Img}\,(\gamma(t))$ is smooth, etc.

$$\int_C F \cdot ds = f(b) - f(a) = 8 - 1 = 7.$$