Solution Set 10D

Math 23a
January 6, 2003

7. First we show that $1/g$ is continuous at $\mathbf{a}$. Fix $\epsilon > 0$, and suppose that $g(\mathbf{a}) = r \neq 0$. Find a $\delta_1 > 0$ such that $|g(\mathbf{b}) - g(\mathbf{a})| < |r|/2$ when $|\mathbf{b} - \mathbf{a}| < \delta_1$, so that $r^2/2 < g(\mathbf{a})g(\mathbf{b}) < 3r^2/2$ for such a $\mathbf{b}$ (note that all quantities are positive). Find a $\delta_2 > 0$ such that $|g(\mathbf{b}) - g(\mathbf{a})| < \epsilon \cdot r^2/2$ when $|\mathbf{b} - \mathbf{a}| < \delta_2$. Set $\delta = \min(\delta_1, \delta_2)$, so if $|\mathbf{b} - \mathbf{a}| < \delta$, we have

$$\frac{1}{g(\mathbf{b})} - \frac{1}{g(\mathbf{a})} = \frac{g(\mathbf{a}) - g(\mathbf{b})}{g(\mathbf{a})g(\mathbf{b})} < \frac{\epsilon \cdot r^2/2}{r^2/2} = \epsilon.$$

Therefore $1/g$ is continuous at $\mathbf{a}$.

Now we will show that in general, if f and h are continuous at $\mathbf{a}$ then so is $f \cdot h$ (so that $f/g = f \cdot (1/g)$ is continuous at $\mathbf{a}$). Fix $\epsilon > 0$, and choose $\gamma > 0$ such that $|f(\mathbf{b}) - f(\mathbf{a})| < 1$ when $|\mathbf{b} - \mathbf{a}| < \gamma$, so that $|f(\mathbf{b})| < |f(\mathbf{a})| + 1$ for such $\mathbf{b}$. Choose $\delta_1 > 0$ such that $(|f(\mathbf{a})| + 1) \cdot |h(\mathbf{b}) - h(\mathbf{a})| < \epsilon/2$ when $|\mathbf{b} - \mathbf{a}| < \delta_1$. Similarly, choose $\delta_2 > 0$ such that $|h(\mathbf{a})| \cdot |f(\mathbf{a}) - f(\mathbf{b})| < \epsilon/2$ when $|\mathbf{b} - \mathbf{a}| < \delta_2$, and set $\delta = \min(\delta_1, \delta_2, \gamma)$. Then for $|\mathbf{b} - \mathbf{a}| < \delta$, we have

$$\begin{aligned} |f(\mathbf{b})h(\mathbf{b}) - f(\mathbf{a})h(\mathbf{a})| &= |f(\mathbf{b})(h(\mathbf{b}) - h(\mathbf{a})) - h(\mathbf{a})(f(\mathbf{a}) - f(\mathbf{b}))| \\ &\leq |f(\mathbf{b})| \cdot |h(\mathbf{b}) - h(\mathbf{a})| + |h(\mathbf{a})| \cdot |f(\mathbf{a}) - f(\mathbf{b})| \\ &< (|f(\mathbf{a})| + 1) \cdot |h(\mathbf{b}) - h(\mathbf{a})| + |h(\mathbf{a})| \cdot |f(\mathbf{a}) - f(\mathbf{b})| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

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8. a) The (partial) graph is given in Figure 8.1.

b) Let $a \in [0, 1]$ be rational. In order to show that f is not continuous at a, we have to show that there is some $\epsilon > 0$ such that for any $\delta > 0$, there exists an $x \in [0, 1]$ such that $|a - x| < \delta$ but $|f(a) - f(x)| \geq \epsilon$. Suppose that $a = p/q$ in lowest form, so that $f(a) = 1/q$. Set $\epsilon = 1/q > 0$, and choose any $\delta > 0$. We can find a large enough integer n such that $\pi/n < \delta$, so that if $x = a + \pi/n$ then $|a - x| < \delta$. Clearly x is irrational because π is irrational (i.e., if x were rational then so would $\pi = n(x - a)$ because $\mathbf{Q}$ is a field). Therefore $f(x) = 0$, so $|f(a) - f(x)| = \epsilon \geq \epsilon$. Thus f is not continuous at a.

c) For $n \in \mathbf{N}$, define the set

$$\begin{aligned} Q_n &= \{a \in [0, 1] \;:\; a \text{ can be written in lowest terms as a fraction } p/q \text{ with } q \leq n\} \\ &= f^{-1}([1/n, \infty)). \end{aligned}$$

Clearly each Q_n is a finite set. Let $r \in [0, 1]$ be irrational, and fix $\epsilon > 0$. Find an integer $n > 0$ such that $1/n < \epsilon$. Set $\delta = \min\{|x - r| \;:\; x \in Q_n\}$ and note that $\delta \neq 0$, so if $y \in [0, 1]$ such that $|y - r| < \delta$ then $y \notin Q_n = f^{-1}([1/n, \infty))$ so that $|f(y) - f(r)| = f(y) < 1/n < \epsilon$. Therefore f is continuous at r.

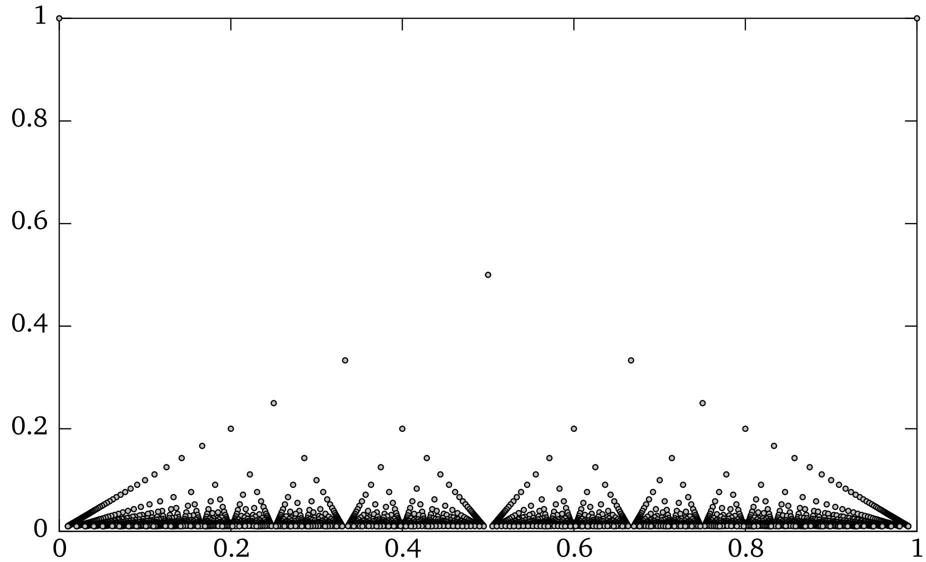

Figure 8.1: The graph of $f(x)$.

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