MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 1
Due: September 21, 2001

1. Read the Appendix and Section 1.1 of Edwards.

2. Verify that the operations addition and multiplication are well-defined for rational numbers, as defined by equivalence classes of ordered pairs of integers.

3. Considering the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers, name the equivalence class that acts as the multiplicative identity, and verify that it does.

4. Considering the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers, prove the existence of multiplicative inverses (for elements other than the additive identity).

5. Show that the field $\mathbb{Z}/2\mathbb{Z}$ is not an ordered field. That is, show that there is no possible choice for a set P of positive elements such that axioms P1–P3 hold.

6. Show that additive inverses in fields are unique. That is, show that given $a \in F$, there exists a *unique* element $b \in F$ such that $a + b = 0$.

7. Use the definition of limits to show that the sequence of real numbers $\left\{ \sin\left(\dfrac{1}{n}\right) \right\}_{n=1}^{\infty}$ converges to 0.

8. Use the definition of Cauchy sequence to show that the sequence of rational numbers $\left\{ \dfrac{1}{n^2} \right\}_{n=1}^{\infty}$ is a Cauchy sequence.

9. Let x be a real number. Show that there is an integer k such that x may be represented in the form

$$x = \sum_{i=k}^{\infty} a_i \cdot 10^{-i} = a_k \cdot 10^{-k} + a_{k-1} \cdot 10^{1-k} + \cdots + a_{-1} \cdot 10^{1} + a_0 + a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + \cdots$$

where $a_i \in \{0, 1, 2, \ldots, 9\}$ for every $i \geq 1$. Show that this representation is unique, except in the case where there exists some $n \in \mathbb{N}$ such that one such representation has $a_i = 0$ for all $i \geq n$ and one such representation has $a_i = 9$ for all $i \geq n$.

10. In the following exercise, we begin a construction of a different completion of the rational numbers $\mathbb{Q}$. The eventual result is the 2-adic field $\mathbb{Q}_2$ instead of the usual real numbers $\mathbb{R}$.

For any non-zero $x \in \mathbb{Q}$, write $x = 2^n \cdot \frac{a}{b}$, where $\frac{a}{b}$ is in lowest terms (that is, a and b have a greatest common divisor of 1) and 2 divides neither a nor b. Define a new absolute value $|\cdot|_2$ on $\mathbb{Q}$ as follows:

$$|x|_2 = 2^{-n}$$

We also define $|0|_2 = 0$.

(a) Show that this absolute value satisfies the usual rules:
   i. $|x|_2 > 0$, for all non-zero $x \in \mathbb{Q}$
   ii. $|xy|_2 = |x|_2 \cdot |y|_2$, for all $x, y \in \mathbb{Q}$
   iii. $|x + y|_2 \leq |x|_2 + |y|_2$, for all $x, y \in \mathbb{Q}$

(b) Show that not only is the Triangle Inequality valid for this absolute value, but the following Ultra-metric Inequality is also valid:
   iv. $|x + y|_2 \leq \max\{|x|_2, |y|_2\}$, for all $x, y \in \mathbb{Q}$