MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 1
Due: February 7, 2003

Homework Assignment #1 (Final Version)

1. Read Fitzpatrick, Sections 11.2, 11.3, 12.5 and Chapter 13.

2. (A) Let $A \subset \mathbb{R}^n$ be compact, and suppose that $B \subset A$ is closed. Use the "open cover" definition to show that B is compact.

3. (B) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, and let $A \subset \mathbb{R}^n$ be compact. Use the "open cover" definition to show that $f(A)$ is compact.

4. (C) The Cantor Intersection Theorem states that if $\{Q_n\}_{n=1}^{\infty}$ is a nested (so that $Q_{n+1} \subset Q_n, \forall n$) collection of non-empty, bounded, closed sets in $\mathbb{R}^n$, then $S = \bigcap_{n=1}^{\infty} Q_n$ is also non-empty, bounded, and closed. Illustrate that the hypotheses are necessary by giving examples of the following cases:

   (a) the Q_n are nested, non-empty, and bounded, but not closed, and $\bigcap_{n=1}^{\infty} Q_n = \emptyset$.

   (b) the Q_n are nested, non-empty, and closed, but not bounded, and $\bigcap_{n=1}^{\infty} Q_n = \emptyset$.

5. (D) Let $V = \mathbb{R}^n$. Recall that we define the collection of linear transformations (and their matrices with respect to the standard basis) from V to V to be:
$$M_n(\mathbb{R}) = \{A : V \longrightarrow V \mid A \text{ is linear}\},$$
and that we may assume that $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ has the structure of Euclidean space. We also define the collection of invertible linear transformations, called the *general linear group*, to be:

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A \text{ is invertible}\}.$$

Consider the following collection of matrices, called the *orthogonal group*:

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid A^t A = I\},$$

(Recall that A^t is the transpose of the matrix A.)

(a) Show that $A \in O_n(\mathbb{R})$ if and only if A is *inner-product preserving*.

Hints:

1. Recall that a linear transformation T is said to be *inner-product preserving* if $\langle T(\mathbf{v}), T(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle, \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
2. Note that the inner-product may be written as follows: $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^t \mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are thought of as $n \times 1$ matrices, in coordinates with respect to the standard basis.

(b) Show that the columns of any $A \in O_n(\mathbb{R})$ form an orthonormal basis for V.

(c) Show that $O_n(\mathbb{R})$ is a *subgroup* of $GL_n(\mathbb{R})$ by showing that if $A, B \in O_n(\mathbb{R})$, then:

i. $AB \in O_n(\mathbb{R})$ (closure of multiplication)
ii. $A^{-1} \in O_n(\mathbb{R})$ (existence of inverses)

(d) If $A \in O_n(\mathbb{R})$, find all possible values of $det(A)$.

(e) Use part (d) and our theorem about connectedness to show that $O_n(\mathbb{R})$ is disconnected.

(f) Show that $O_n(\mathbb{R})$ is compact in $M_n(\mathbb{R})$ by showing that:

i. $O_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.
ii. $O_n(\mathbb{R})$ is bounded in $M_n(\mathbb{R})$. *(Hint: Use part (b).)*