Math 23b, Spring 2003

Problem Set 1, Part B
Solutions written by Tseno Tselkov

Problem 3: *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, and let $A \subset \mathbb{R}^n$ be compact. Use the "open cover" definition to show that $f(A)$ is compact.*

Proof. Take an open cover $\{U_i\}_{i\in I}$ of $f(A)$ (i.e. each U_i is open and $f(A) \subseteq \bigcup_{i\in I} U_i$; I is some index set). To show that $f(A)$ is compact we need to find a finite subcover. But since f is continuous we have that each $f^{-1}(U_i)$ is open in $\mathbb{R}^n$. Thus since $\{U_i\}_{i\in I}$ is an open cover of $f(A)$, then $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of A. But A is compact $\Rightarrow$ every open cover of A has a finite subcover. Applying this to the particular open cover of A from above we obtain a finite subcover $\{f^{-1}(U_i)\}_{i\in J}$ of A (J is a finite subset of I). But since $\{f^{-1}(U_i)\}_{i\in J}$ covers A we directly get that $\{U_i\}_{i\in J}$ covers $f(A)$. Thus we obtained a finite subcover of $f(A) \Rightarrow f(A)$ is compact and we're done. $\square$