If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

4

When you are asked for counter-examples, it's best to produce the simplest possible counter-example. There's nothing wrong with an a counter-example that generalizes to $\mathbb{R}^n$, but it might confuse the point.

The Cantor Intersection Theorem states that if $\{Q_n\}$ is a collection of nested, closed, and bounded subsets of $\mathbb{R}^n$, then their intersection $S = \bigcap_1^\infty Q_n$ is non-empty. In part (a), we show that the closed condition is necessary. In part (b), we show that the bounded condition is necessary.

1. Consider $Q_n = (0, 1/n) \subset \mathbb{R}^1$, $n \in \mathbb{N}$. Q_n is not closed, because 0 is a limit point of Q_n, but it is not contained in Q_n. Q_n is bounded because $Q_n \subset [0, 1]$. Furthermore, $Q_{n-1} \subset Q_n$ implies that the Q_n are nested subsets.

   Suppose $x \in S$, the intersection of all the Q_n. Since $x \in Q_1$, $x > 0$. We can find an $n \in \mathbb{N}$ such that $1/n < x$. Hence, $x \notin Q_n$, which implies $x \notin S$. This shows that the intersection is empty.

2. Consider $Q_n = [n, \infty)$. (It is notationally traditional to use the open-parentheses after the infinity; it is like writing $n \leq x < \infty$ as opposed to $n \leq x \leq \infty$.) Q_n is closed (beacuse it's complement is open). The Q_n are nested.

   Consider $x \in \mathbb{R}$. $n = \lceil x \rceil + 1$, an integer strictly greater than x. $x \notin Q_n$, so $x \notin S$.