Solution Set 1D

Math 23b
February 13, 2003

5. a) **Claim.** *$A \in \mathrm{O}_n(\mathbf{R})$ iff A is inner-product preserving.*

Proof.

($\Longrightarrow$) Let $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$, and let $A \in \mathrm{O}_n(\mathbf{R})$, so $A^t A = I$. Then

$$\langle A\mathbf{v}, A\mathbf{w} \rangle = (A\mathbf{v})^t (A\mathbf{w}) = \mathbf{v}^t A^t A \mathbf{w} = \mathbf{v}^t I \mathbf{w} = \langle \mathbf{v}, \mathbf{w} \rangle .$$

($\Longleftarrow$) If A is inner-product preserving then for any $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$,

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle = \langle A\mathbf{v}, A\mathbf{w} \rangle = (A\mathbf{v})^t (A\mathbf{w}) &= \mathbf{v}^t (A^t A \mathbf{w}) = \left\langle \mathbf{v}, A^t A \mathbf{w} \right\rangle \\ &\Longrightarrow \left\langle \mathbf{v}, (I - A^t A)\mathbf{w} \right\rangle = 0. \end{aligned}$$

Suppose that $I - A^t A \neq 0$. Then there is some $\mathbf{w} \in \mathbf{R}^n$ such that $(I - A^t A)\mathbf{w} \neq 0$. Set $\mathbf{v} = (I - A^t A)\mathbf{w}$, so by positive-definiteness, we have $\langle \mathbf{v}, (I - A^t A)\mathbf{w} \rangle \neq 0$, a contradiction. Therefore $A^t A = I$.

❐

b) Let $A \in \mathrm{O}_n(\mathbf{R})$, and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard coordinate vectors in $\mathbf{R}^n$. Then the columns of A are given by $A\mathbf{e}_1, \ldots, A\mathbf{e}_n$. Since A is inner-product preserving, we have

$$\langle A\mathbf{e}_i, A\mathbf{e}_j \rangle = \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$$

(where δ is the Kronecker delta) — that is, the columns of A are orthonormal (and therefore linearly independent). Since there are n such columns, they form a basis.

Note that what we have proved is equivalent to the statement that orthogonal matrices take orthonormal bases to orthonormal bases.

c) (i) Suppose that $A, B \in \mathrm{O}_n(\mathbf{R})$, so A and B are inner-product preserving. Then for any $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$,

$$\langle AB\mathbf{v}, AB\mathbf{w} \rangle = \langle B\mathbf{v}, B\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$$

so AB is inner product preserving, and therefore $AB \in \mathrm{O}_n(\mathbf{R})$.

(ii) Let $A \in \mathrm{O}_n(\mathbf{R})$, so A is inner-product preserving. Then for any $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$,

$$\left\langle A^{-1}\mathbf{v}, A^{-1}\mathbf{w} \right\rangle = \left\langle AA^{-1}\mathbf{v}, AA^{-1}\mathbf{w} \right\rangle = \langle \mathbf{v}, \mathbf{w} \rangle$$

so $A^{-1} \in \mathrm{O}_n(\mathbf{R})$. Note that $A^{-1} = A^t$.

d) If $A \in \mathrm{O}_n(\mathbf{R})$ then $A^t A = I$, so since $\det(A) = \det(A^t)$, we have

$$1 = \det(I) = \det(A^t A) = \det(A^t) \det(A) = \det(A)^2$$

so $\det(A) = \pm 1$. Note that $I \in \mathrm{O}_n(\mathbf{R})$ and $\det(I) = 1$, and that

$$\begin{bmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \in \mathrm{O}_n(\mathbf{R})$$

has determinant -1, so that both values are attained.

e) We know that $\det : \mathrm{O}_n(\mathbf{R}) \to \{\pm 1\}$ is continuous and surjective. If $\mathrm{O}_n(\mathbf{R})$ is connected then $\{\pm 1\} = \det(\mathrm{O}_n(\mathbf{R}))$ is connected; therefore it suffices to show that $\{\pm 1\}$ is disconnected. This is clear because $(-\infty, 0)$ and $(0, \infty)$ are an open cover of $\{\pm 1\}$.

f) We can show that $\mathrm{O}_n(\mathbf{R})$ is compact using Heine-Borel.

(i) Consider the map $f : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ given by $f(A) = A^t A$. If one writes out the matrix multiplication, one finds that the entries of $A^t A$ are all polynomials in the entries of A, so since coordinate projections, sums, and products are all continuous, the function f is also continuous. By definition, $\mathrm{O}_n(\mathbf{R}) = f^{-1}(\{I\})$, so since $\{I\}$ is closed (because it is a one-point set), we have that $\mathrm{O}_n(\mathbf{R})$ is closed as well.

(ii) If $A \in \mathrm{O}_n(\mathbf{R})$ and a_{ij} are the entries of A then by definition,

$$\|A\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = \sum_{i=1}^{n} \langle A\mathbf{e}_i, A\mathbf{e}_i \rangle = n$$

by (b). Thus $A \in B_{\sqrt{n}+1}(I)$, so since A was arbitrary, we have shown that $\mathrm{O}_n(\mathbf{R})$ is bounded.

Notes on this problem:

(1) The statement "$\mathbf{v}^t A^t A \mathbf{w} = \mathbf{v}^t \mathbf{w} \implies A^t A = I$" must be justified; it is not immediately obvious. It is only true because $\mathbf{v}^t A^t A \mathbf{w} = \mathbf{v}^t \mathbf{w}$ *for all* $\mathbf{v}$ and $\mathbf{w}$; a single pair does not suffice. And then you have to somehow or other use the nondegeneracy of the standard inner product on $\mathbf{R}^n$; the statement is not true for all inner products.

Be careful what space your variables live in! Many people tried to "cancel the $\mathbf{w}$" by multiplying by $\mathbf{w}^t$ on both sides. That's fine, but $\mathbf{w}\mathbf{w}^t$ is an $n \times n$ matrix, which probably isn't what you wanted!

(2) The reason that $\mathrm{O}_n(\mathbf{R})$ is closed is that the map $B \mapsto B^t B$ is continuous, and $\{I\}$ is closed. You probably lost a point or two if I wasn't convinced that you understood that this is the key fact.

(3) A common error was to assert that $\mathrm{O}_n(\mathbf{R}) = \det^{-1}(\{\pm 1\})$. This is false; certainly $\mathrm{O}_n(\mathbf{R}) \subset \det^{-1}(\{\pm 1\})$, but equality does not hold. Consider the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbf{R})$. This clearly has determinant one but is not orthogonal. It is true that the inverse image of $\{\pm 1\}$ under $\det$ is a closed subgroup of $\mathrm{GL}_n(\mathbf{R})$, but this is not the subgroup that you want.

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