Math 23a, 2002.
Problem Set 1, Question 4.

Question. Considering the intergers as defined by equivalence classes of pairs of natural numbers, prove the existence of additive intergers. *(Hint: To do this, you will need first to name the additive identity correctly.)*

Answer. To start with, let's consider the hint. A lot of people were able instictively to identify the additive identity as $[(1,1)]$. If you didn't see it, you can still construct it. Let $[(a,b)]$, $[(n,m)] \in \mathbb{Z}$, for a, b, n, and $m \in \mathbb{N}$ where $[(n,m)]$ is the additive identity. Then we have

$$[(a,b)] + [(n,m)] = [(a+n, b+m)]$$

thanks to the definition of interger addition. If $[(m,n)]$ is an identity, the right hand side equals $[(a,b)]$ and

$$a + (b+m) = b + (a+n)$$

by our equivalence relation $(a,b) \sim (c,d)$ if and only if $a+d = b+c$. And since addition over the natural numbers is associative and commutative,

$$(a+b) + m = (a+b) + n.$$

Which gives us back a new equivalence: $(n,m) \sim (a+b, a+b)$.

Now newly equipped with an identity element, we quickly see that for any $x = [(a,b)] \in \mathbb{Z}$ that:

$$[(a,b)] + [(b,a)] = [(b+a, a+b)] = [(a+b, a+b)] = [(a+b, b+a)] = [(b,a)] + [(a,b)].$$

Therefore $[(b,a)] = -x$ is the additive inverse of $x = [(a,b)]$. And since a, $b \in \mathbb{N}$ (else $x \notin \mathbb{Z}$), $-x \in \mathbb{Z}$.