Solution Set 2A

Math 23b
February 19, 2003

1. a) Define $p : \mathbf{R}^2 \to \mathbf{R}$ by $p(s,t) = \|f(s) - g(t)\|^2 = (f(s) - g(t)) \cdot (f(s) - g(t))$. Then if (s_0, t_0) is an absolute minimum for p, we must have $\partial p/\partial s = \partial p/\partial t = 0$. It is obvious that the product rule holds for dot products as well as normal products, so this gives

$$\begin{aligned} 0 &= \frac{\partial p}{\partial t}\bigg|_{(s_0,t_0)} \\ &= \frac{\partial}{\partial t}(f(s) - g(t)) \cdot (f(s) - g(t))\big|_{(s_0,t_0)} \\ &= (f(s_0) - g(t_0)) \cdot f'(s_0) + f'(s_0) \cdot (f(s_0) - g(t_0)) \\ &= 2(\mathbf{p} - \mathbf{q}) \cdot f'(s_0). \end{aligned}$$

The same calculation shows that $(\mathbf{p} - \mathbf{q}) \cdot g'(t_0) = 0$.

b) Let $f(s) = (s, 2s, -s)$ and $g(t) = (t+1, t-2, 2t+3)$. We would like to find an (s_0, t_0) such that $(f(s_0) - g(t_0)) \cdot f'(s_0) = (f(s_0) - g(t_0)) \cdot g'(t_0) = 0$. We have $f'(s) = (1, 2, -1)$ and $g'(t) = (1, 1, 2)$, so

$$\begin{aligned} 0 = (f(s) - g(t)) \cdot f'(s) &= s + 4s + s - t - 1 - 2t + 4 + 2t + 3 = 6s - t + 6 \\ 0 = (f(s) - g(t)) \cdot g'(t) &= s + 2s - 2s - t - 1 - t + 2 - 4t - 6 = s - 6t - 5. \end{aligned}$$

Solving this system of equations, we have

$$(s_0, t_0) = \left(-\frac{41}{35}, -\frac{36}{35}\right)$$

which means that the closest points are

$$\mathbf{p} = \left(-\frac{41}{35}, -\frac{82}{35}, \frac{41}{35}\right) \qquad \mathbf{q} = \left(-\frac{1}{35}, -\frac{106}{35}, \frac{33}{35}\right).$$

Note on this problem:

(1) A lot of people wrote something like $p' = 2(f+g)(f'+g')$. You should really consider what this means — what is p'? It is a linear map $\mathbf{R}^2 \to \mathbf{R}$, or alternatively, $p' = (\partial p/\partial s, \partial p/\partial t)$. It is not a scalar, and it doesn't even make sense to use the product rule. If $2(f+g)(f'+g')$ is to have any interpretation, it is as a dot product, in which case it is definitely not equal to p'. I think that the people who made this mistake somehow were trying to take both partial derivatives at the same time, which doesn't really make much sense either. So the moral is: always keep in mind what spaces your objects live in! Be careful!

On a related note, everyone who made the above mistake then needed to say that $\langle \mathbf{p} - \mathbf{q}, f' - g' \rangle = 0 \implies \langle \mathbf{p} - \mathbf{q}, f' \rangle = \langle \mathbf{p} - \mathbf{q}, g' \rangle = 0$. This is very false (what's a counterexample?). By this reasoning, f' and g' would both be orthogonal to $\mathbf{p}$ and $\mathbf{q}$ as well!

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